B1 Set Theory: Problem sheet 3

1. Show by induction that, given $n \in \omega$, every subset of n is equinumerous with some natural number. Hence prove that any subset of a finite set is finite. (A set a is defined to be finite if a is equinumerous with n for some $n \in \omega$.)

2. Prove that the following properties of a set X are equivalent:

(i) $\omega \preceq X$ (that is, there exists an injective function $f : \omega \to X$),

(ii) there exists a function $g : X \to X$ which is injective but not surjective.

[Hint: for (ii)$\Rightarrow$(i) use the Recursion Theorem. You should use induction to show that the function f you define is indeed injective.]

3. Suppose that κ, λ and μ are cardinals. Prove the following:

(i) $(\kappa + \lambda) + \mu = \kappa + (\lambda + \mu)$,

(ii) $\kappa.(\lambda.\mu) = (\kappa.\lambda).\mu$,

(iii) $\kappa(\lambda + \mu) = \kappa\lambda + \kappa\mu$,

(iv) $\kappa^{\lambda+\mu} = \kappa^{\lambda}\kappa^{\mu}$,

(v) $\kappa^{\lambda\mu} = (\kappa^{\lambda})^{\mu}$,

(vi) $(\kappa\lambda)^{\mu} = \kappa^{\mu}\lambda^{\mu}$.

4. (i) Let A, X and Y be sets such that $X \preceq A$. Prove that ${}^{Y}X \preceq {}^{Y}A$. Deduce that, for cardinals κ, λ and μ, if $\kappa \leq \lambda$, then $\kappa^{\mu} \leq \lambda^{\mu}$.

(ii) Now let A, B, X, Y be sets with the property that $X \preceq A$ and $Y \preceq B$. Prove that, in most cases, ${}^{Y}X \preceq {}^{B}A$ [you should show that the map from ${}^{Y}X$ into ${}^{B}A$ that you set up really is an injection]. What are the exceptional cases?

5. Calculate the cardinalities of the following sets. [*Try to simplify your answers as much as possible; for example,* $\aleph_0$, $2^{\aleph_0}$ *or* $2^{2^{\aleph_0}}$ *is a better answer than, say,* $\aleph_0.(2^{\aleph_0})^{\aleph_0}$.]

(i) The set of all finite sequences of natural numbers. [*Hint: use unique factorisations of natural numbers into powers of primes. Note that we cannot, on the basis of the axioms so far given, prove that a countable union of countable sets is countable.*]

(ii) The set of functions from $\mathbb{R}$ to $\mathbb{R}$.

(iii) The set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. [*Hint: a continuous function is determined by its restriction to* $\mathbb{Q}$.]

(iv) The set of all equivalence relations on ω. [*Hint: to get a lower bound on the cardinality, think in terms of partitions of* ω.]

6. Let $f : X \to Y$ be a surjective function. Prove that $\wp(Y) \preceq \wp(X)$. [*You should not assume that there exists an injective function* $g : Y \to X$*; the axioms so far defined do not allow one to prove this.*]

7. Let κ be any cardinal number.

(i) Prove that $\kappa + 0 = \kappa$, and $\kappa.0 = 0$.

(ii) Prove that for any natural number n, $\kappa.n^+ = \kappa.n + \kappa$.

(iii) For natural numbers, we now have two definitions of addition and multiplication: one derived from the recursive definitions, the other from the theory of cardinal arithmetic.

Prove that these two operations are the same.