

B1 Set Theory: Problem sheet 1

For this sheet, assume the Empty Set Axiom, the Axioms of Extensionality, Pairs, and Unions, and the Subset Axiom Scheme. For question 1., assume that if X is a set, then so is its powerset $\wp X$ (that is, the set of all subsets of X). For question 7., ZFC is the standard axiomatisation of Set Theory; and it is a consequence of the Axiom of Foundation that there does not exist an infinite sequence $x_0, x_1, x_2 \dots$ of sets such that $x_0 \ni x_1 \ni x_2 \ni \dots$

1. Give proofs or counterexamples as appropriate.

(i) $a \setminus \bigcap b = \bigcup \{a \setminus x : x \in b\},$

(ii) $\left(\bigcap a\right) \cup \left(\bigcap b\right) = \bigcap \{x \cup y : x \in a, y \in b\},$

(iii) $\wp\left(\bigcup X\right) = X,$

(iv) $\bigcup \wp(X) = X,$

(v) $\wp(a) \subseteq \wp(b)$ implies $a \subseteq b.$

(Whenever an intersection $\bigcap c$ arises, take the set c to be non-empty.)

2. A set a is called *transitive* if, for all sets x , $x \in a$ implies $x \subseteq a$. Prove that

(i) $\emptyset$ is transitive;

(ii) if a is transitive then so is $a \cup \{a\};$

(iii) a is transitive if and only if $\bigcup(a \cup \{a\}) = a.$

3. (i) Assume that W is the set of all one-element sets. What is $\bigcup W$?

(ii) Prove that W cannot exist.

(iii) Prove that there cannot exist a set having all two-element sets as elements.

4. (i) Prove that if a, b and c are sets, then so is $\{a, b, c\}.$

(ii) Prove that if $x_1, x_2, \dots, x_n$ are sets, then $\{x_1, x_2, \dots, x_n\}$ is a set.

(iii) Prove that if $X = \{x_1, \dots, x_n\}$ is a finite set, then $\wp X$ is a set.

(iv) Prove that if X is a finite set, then the class of all two-element subsets of X is a set.

(v) Assume that the class of natural numbers exists as a set. (In this course, for slightly weird historical reasons, we refer to the set of natural numbers as ω , and assume that $0 \in \omega$.)

Prove that the set of even numbers exists (ie. that the class of even numbers is a set).

5. Prove that there exist infinitely many sets.

6. We say that a set X is *hereditarily finite* iff X is finite, and all elements of X are hereditarily finite.

(i) Prove that the following sets are hereditarily finite:

$$\emptyset, \{\emptyset\}, \{\{\emptyset\}, \emptyset\}, \{\{\{\emptyset\}, \emptyset\}, \{\emptyset\}\}.$$

(ii) Let H be the class of hereditarily finite sets. Prove that the Empty Set Axiom, the Axioms of Extensionality, Pairs, and Unions, and the Subset Axiom Scheme are all true in H . [For instance, the Axiom of Pairs is true in H iff whenever a and b are hereditarily finite sets, then there is a hereditarily finite set c whose only hereditarily finite elements are a and b . This will be true if $\{a, b\}$ is a hereditarily finite set.]

(iii) Is the set of natural numbers an element of H ?

(iv) Show that it is impossible to prove that the set of natural numbers exists from the Empty Set Axiom, the Axioms of Extensionality, Pairs, and Unions, and the Subset Axiom Scheme.

(v) Let K be the set of hereditarily countable sets. (We define a set to be *hereditarily countable* iff it is countable, and all its elements are hereditarily countable. As is the case for hereditarily finite sets, this definition does make sense.)

Assume that $\omega \in K$. Which of the Empty Set Axiom, The Axiom of Extensionality, the Axiom of Pairs, the Axiom of Unions, and the Subset Axiom Scheme are true in K ? [You may assume that a countable union of countable sets is countable, even though this does not follow from the axioms so far given.]

Is it possible to prove, from the axioms mentioned in the previous paragraph, that for every set x , its powerset $\wp x$ is a set?

7. (Optional) The following argument claims to show that ZFC, the system of first order axioms of Set Theory, is inconsistent. Is there a mistake in the argument, and if so where?

Assume ZFC.

Now ZFC is a consistent set of sentences in a first order language, so by the Completeness Theorem (which follows from ZFC), there exists a set $\mathfrak{M}_0$ such that $\mathfrak{M}_0$ is a model of ZFC.

Now since $\mathfrak{M}_0$ is a model of ZFC, the Completeness Theorem is true inside $\mathfrak{M}_0$, so there is an element $\mathfrak{M}_1$ of $\mathfrak{M}_0$ such that $\mathfrak{M}_1$ is a model of ZFC. Continuing recursively, we construct a sequence $\mathfrak{M}_0 \ni \mathfrak{M}_1 \ni \mathfrak{M}_2 \ni \dots$, contradicting the Axiom of Foundation.

This contradiction shows that ZFC is inconsistent.