MFoCS questions should be done by MFoCS students, although everyone is encouraged to try them; we may not have time to go through them in classes. Questions (or parts of questions) marked with a + sign are intended as a challenge for enthusiasts: we will not go through them in classes!

1. What are the 50th, 51st and 52nd elements of $\mathbb{N}^{(3)}$ in the colex order? What about in lex?

2. Let $\mathcal{F} \subset [10]^{(3)}$, and suppose $|\mathcal{F}| = 29$.

   (a) What is the minimum possible size of $\partial\mathcal{F}$?

   (b) Find a family that achieves this minimum.

3. Suppose that $\mathcal{F} \subset [n]^{(r)}$, and let $\mathcal{A}$ denote the first $|\mathcal{F}|$ elements of $[n]^{(r)}$ in colex order. If $|\partial F| = |\partial A|$ must we have $\mathcal{F} = \mathcal{A}$? [You should give an answer for every $r \geq 2$.]

4. The *upper shadow* $\partial^+(\mathcal{F})$ of a set $\mathcal{F} \subset [n]^{(r)}$ is the set

$$\partial^+(\mathcal{F}) := \{A \in [n]^{(r+1)} : A \supset B \text{ for some } B \in \mathcal{F}\}.$$

   Give a version of the Kruskal-Katona Theorem for the upper shadow.

5. Give a proof of Hall's Theorem using Dilworth's Theorem.

6. Prove that in any sequence of n^2+1 real numbers there is an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$. [Bonus: try to find more than one proof.]

7. We say that $\mathcal{A} \subset \mathcal{P}(n)$ is a *downset* if, for every $A \in \mathcal{A}$, every subset of A belongs to $\mathcal{A}$. Prove that if $\mathcal{A}$ is a downset then the average size of sets in $\mathcal{A}$ is at most $n/2$.

8. Prove that every intersecting family $\mathcal{F} \subset \mathcal{P}(n)$ is contained in an intersecting family of size 2^{n-1}.

9. Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be subsets of $[n]$ such that $|A_i \cap B_i|$ is odd for all i and $|A_i \cap B_j|$ is even for all $i \neq j$. Show that $m \leq n$. [*Hint:* Consider the characteristic vectors of the sets A_i. We will prove this result later in the course, but try to do it without looking ahead!]

10. (MFoCS) Let A be an m by n matrix with all entries in $\{0, 1\}$. We say that A is *row-lexicographic* if the rows $r_1, \ldots, r_m$ of A (considered as words of length n) are in lexicographic order, and that A is *column-lexicographic* if the columns $c_1, \ldots, c_m$ of A are in lexicographic order. It is always possible to permute the rows or columns of a matrix so that it is (respectively) row-lexicographic or column-lexicographic.

    Is it always possible to permute the rows of A, and then the columns of the resulting matrix, to obtain a matrix that is *both* row-lexicographic and column-lexicographic?

11.$^+$ Suppose that $\mathcal{A} \subset \mathcal{P}(n)$ does not contain distinct sets A_1, A_2, A_3 with $A_1 \cup A_2 = A_3$. Prove that $|\mathcal{A}| \leq C\binom{n}{\lfloor n/2 \rfloor}$ for some absolute constant $C > 0$.