MFoCS questions should be done by MFoCS students, although everyone is encouraged to try them; we may not have time to go through them in classes. Questions (or parts of questions) marked with a + sign are intended as a challenge for enthusiasts: we will not go through them in classes!

1. Let $\mathcal{A} \subset \mathcal{P}(n)$ be an upset and $\mathcal{B} \subset \mathcal{P}(n)$ be a downset. Prove that $|\mathcal{A} \cap \mathcal{B}| \leq 2^{-n}|\mathcal{A}| \cdot |\mathcal{B}|$.

2. (a) Let $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ be an intersecting set system. Must there be a finite set A such that $\{F \cap A : F \in \mathcal{F}\}$ is an intersecting system?

   (b) What if $\mathcal{F} \subset \mathbb{N}^{(r)}$?

3. A *sunflower* is a sequence $F_1, \ldots, F_k$ of sets such that for some set S, and all $i < j$,

$$F_i \cap F_j = S.$$

   Let $r, s \geq 1$. Prove that there is $m = m(r, s)$ such that every sequence of m sets from $\mathbb{N}^{(r)}$ has a subsequence of length s that forms a sunflower.

   [Bonus question: explain the term *sunflower* by means of a nice picture.]

4. The *i-compression operator* π_i is defined by $\pi_i(A) = A \setminus \{i\}$ and, for a set system $\mathcal{A}$,

$$\pi_i(\mathcal{A}) = \{\pi_i(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : \pi_i(A) \in \mathcal{A}\}.$$

   Let $\mathcal{F} \subset \mathcal{P}(n)$ be a set system and $\mathcal{A} = \pi_i(\mathcal{F})$ for some $i \in [n]$. Show that $\mathrm{tr}_{\mathcal{A}}(S) \leq \mathrm{tr}_{\mathcal{F}}(S)$ for every $S \subset [n]$.

5. Let $\mathcal{F}$ be the collection of all convex sets in $\mathbb{R}^2$. Show that $\mathcal{F}$ does not have bounded VC-dimension.

6. Let $\mathcal{F} \subset \mathcal{P}(X)$. The *dual system* $\mathcal{F}^*$ has vertex set $\mathcal{F}$, and for each $x \in X$, there is an edge $\{F \in \mathcal{F} : x \in F\}$ (we ignore duplicate edges). Prove that for every positive integer d there is a constant $f(d)$ such that if $\mathcal{F}$ has VC-dimension at most d then $\mathcal{F}^*$ has VC-dimension at most $f(d)$.

7. Suppose that $\mathcal{F}_1, \ldots, \mathcal{F}_s \subset \mathcal{P}(n)$ are intersecting families. Prove that $|\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_s| \leq 2^n - 2^{n-s}$.

8. A function $f : \mathcal{P}(n) \to \mathbb{R}$ is *monotone increasing* if $f(A) \leq f(B)$ whenever $A \subset B$. Prove that if f and g are nonnegative, monotone increasing functions on $\mathcal{P}(n)$ then

$$\sum_{A \subset [n]} f(A)g(A) \geq 2^{-n} \sum_{A \subset [n]} f(A) \sum_{A \subset [n]} g(A).$$

9. (MFoCS) Let $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ be the set system $\{N_k : k \in \mathbb{N}\}$, where $N_k = \{k, 2k, 3k, \ldots\}$. Does $\mathcal{F}$ have bounded VC-dimension?

10.$^+$ Let $\mathcal{F}$ be the collection of all half-spaces in $\mathbb{R}^k$. Show that $\mathcal{F}$ has bounded VC-dimension. Can you determine the VC-dimension exactly?

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