Combinatorics C8.3 **Problems 4** **ADS MT2020**

There are no MFoCs questions this week. Everyone should try everything! Questions (or parts of questions) marked with a + sign are intended as a challenge for enthusiasts: we will not go through them in classes!

1. Show that, for some $c > 1$ and every $n \geq 5$, there is a family $\mathcal{F} \subset \mathcal{P}(n)$ of size at least c^n such that every set in $\mathcal{F}$ has odd size, and the intersection of any two distinct sets from $\mathcal{F}$ has odd size.

2. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(n)$ be two set systems such that $|A \cap B|$ is even for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that $|\mathcal{A}| \cdot |\mathcal{B}| \leq 2^n$. Can you describe the pairs $\mathcal{A}$, $\mathcal{B}$ for which we have equality? [Hint: Show that if $A, A' \in \mathcal{A}$ then we may assume $A \triangle A' \in \mathcal{A}$.]

3. Let P be a set of n points in the plane that do not all lie on a straight line. Prove that they determine at least n lines. [Hint: For each point, consider the set of lines that passes through it.]

4. Prove that a non-trivial decomposition of the edges of K_n into edge-disjoint complete subgraphs requires at least n subgraphs. Show how this bound can be achieved.

5. A set P in $\mathbb{R}^n$ is a *two-distance set* if there are real numbers α, β such that $||x - y||_2 \in \{\alpha, \beta\}$ for all distinct $x, y \in P$. Let $P = \{p_1, \dots, p_k\}$ be a two-distance set.

   (a) For each $i \in [k]$, let f_i be the polynomial in variables $x = (x_1, \dots, x_n)$ defined by
$$f_i(x) = (||x - p_i||_2^2 - \alpha^2)(||x - p_i||_2^2 - \beta^2).$$
   Show that the polynomials f_i are linearly independent. [Hint: Consider $f_i(x_j)$.]

   (b) Deduce that $k \leq \binom{n}{2} + 3n + 2$. [Hint: Find a basis for the space spanned by the polynomials f_i.]

6. Let $\mathcal{F}$ be a collection of functions from $[n]$ to $\mathbb{Z}$. Suppose that, for every pair of distinct functions $f, g \in \mathcal{F}$ we have $f(i) = g(i) + 1$ for some i. Prove that $|\mathcal{F}| \leq 2^n$. [Hint: look for a suitable collection of polynomials.]

7. Let C_n be the cycle of length n, and let $S_1, \ldots, S_n$ be subsets of $\mathbb{R}$ with $|S_i| = 2$ for each i.

(a) Show that if n is even then it is possibly to choose elements $s_i \in S_i$ for each i such that elements chosen for adjacent vertices are distinct. [Hint: Use the Combinatorial Nullstellensatz.]

(b) What happens if n is odd?

8. Prove that there is an uncountable collection $\mathcal{A}$ of subsets of $\mathbb{N}$ such that $|A \cap B|$ is finite for all distinct $A, B \in \mathcal{A}$.

9.$^+$ Let $1 \leq i \leq j \leq n$. Let $A = (A_{ST})$ be a $\binom{n}{i} \times \binom{n}{j}$ matrix with rows indexed by elements of $[n]^{(i)}$ and columns indexed by elements of $[n]^{(j)}$, where $a_{ST} = 1$ if $S \subset T$ and $a_{ST} = 0$ otherwise. Prove that $\text{rank}(A) = \min\{\binom{n}{i}, \binom{n}{j}\}$.

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