MFoCS questions should be done by MFoCS students, although everyone is encouraged to try them; we may not have time to go through them in classes. Questions (or parts of questions) marked with a + sign are intended as a challenge for enthusiasts: we will not go through them in classes!

1. Write down all antichains contained in $\mathcal{P}(1)$ and $\mathcal{P}(2)$. How many different antichains are there in $\mathcal{P}(3)$?

2. Let $k \leq n/2$, and suppose that $\mathcal{F}$ is an antichain in $\mathcal{P}(n)$ such that every $A \in \mathcal{F}$ has $|A| \leq k$. Prove that $|\mathcal{F}| \leq \binom{n}{k}$.

3. Let $(P, \leq)$ be a poset. Suppose that every chain in P has at most k elements. Prove that P can be written as the union of k antichains.

4. Suppose $\mathcal{F} \subset \mathcal{P}(n)$ is a set system containing no chain with $k+1$ sets.

   (a) Prove that
   $$\sum_{i=0}^{n} \frac{|\mathcal{F}_i|}{\binom{n}{i}} \leq k,$$
   where $\mathcal{F}_i = \mathcal{F} \cap [n]^{(i)}$ for each i.

   (b) What is the maximum possible size of such a system?

5. (a) Look up Stirling's Formula. Use it to find an asymptotic estimate of form $(1 + o(1))f(n)$ for $\binom{n}{n/2}$.

   (b) Now do the same for $\binom{n}{pn}$ where $p \in (0,1)$ is a constant. Write your answer in terms of the binary entropy function $H(p) = -p \log p - (1-p)\log(1-p)$.

6. Let $\mathcal{A}$ be an antichain in $\mathcal{P}(n)$ that is not of the form $[n]^{(r)}$. Must there exist a maximal chain disjoint from $\mathcal{A}$?

7. Let $(P, \leq)$ be an infinite poset. Must P contain an infinite chain or antichain?

8. (MFoCS) Let $f(n)$ be the number of subsets $\mathcal{A} \subset \mathcal{P}(n)$ such that $\mathcal{A}$ is an antichain.

(a) Prove that $f(n) \geq 2^{\binom{n}{\lfloor n/2 \rfloor}}$ for every n.

(b) Prove that, for every $\epsilon > 0$, $f(n) \leq 2^{\epsilon 2^n}$ for all sufficiently large n.

(c)$^+$ Prove that there is a constant $C > 0$ such that

$$f(n) \leq C^{\binom{n}{\lfloor n/2 \rfloor}}$$

for every n.

9.$^+$ Consider the poset $P = (\mathcal{P}(\mathbb{N}), \subseteq)$ (in other words, the elements of P are sets of natural numbers, and $A \leq B$ if $A \subseteq B$).

(a) Show that P contains an uncountable antichain.

(b) Does P contain an uncountable chain?

scott@maths.ox.ac.uk