Combinatorics C8.3 **Problems 0** **ADS MT2020**

There will not be classes on this problem sheet, but solutions will be available on the course webpage from the end of Week 0.

Part B Graph Theory is not a prerequisite for the course, but we will use a little standard notation, as well as Hall's Theorem on matchings in bipartite graphs (this can can be found in the Part B Graph Theory notes). All graphs below are assumed to be finite.

1. Let G be a bipartite graph with bipartition (A, B). Suppose that every vertex in G has the same degree $d > 0$.
   (a) Show that $|A| = |B|$.
   (b) Look up Hall's theorem. Use this result to prove that G contains a complete matching.
   (c) Show that the edge set of G can be partitioned into d edge disjoint complete matchings.

2. Let $\mathcal{P}(n)$ denote the power set of $[n] := \{1, \ldots n\}$. For $A, B \in \mathcal{P}(n)$, we define the *symmetric difference of A and B* is $A \triangle B := (A \setminus B) \cup (B \setminus A)$.
   (a) Suppose $\mathcal{A} \subset \mathcal{P}(n)$, and there do not exist $A, B \in \mathcal{A}$ with $|A \triangle B| = 1$. How large can $|\mathcal{A}|$ be?
   (b) For $n \geq 1$, give two examples of $\mathcal{A}$ with maximal size. Are there any others?

3. Let $[n]^{(i)} := \big\{A \subset \{1, \ldots, n\} : |A| = i\big\}$ and suppose that $i \leq n/2$. Prove that there is a bijection $f : [n]^{(i)} \to [n]^{(i)}$ such that $A \cap f(A) = \emptyset$ for every A.

4. (a) Prove that $|\mathcal{P}(n)| = 2^n$.
   (b) Suppose a set $A \in \mathcal{P}[n]$ is selected uniformly at random. Let X denote the random variable given by $X(A) := |A|$. Prove that $\mathbb{E}(X) = n/2$ and $\mathrm{var}(X) = n/4$.
   (c) Use Chebyshev's inequality and (b) to show that given $\epsilon > 0$ there is $C > 0$ such that at least $(1 - \epsilon)2^n$ sets $A \subset [n]$ satisfy $\big||A| - \frac{n}{2}\big| \leq C\sqrt{n}$.