Combinatorics C8.3 **Problem 0 solutions** **ADS MT2020**

1. Let G be a bipartite graph with bipartition (A, B). Suppose that every vertex in G has the same degree $d > 0$.

   (a) Show that $|A| = |B|$.

   (b) Look up Hall's theorem. Use this result to prove that G contains a complete matching.

   (c) Show that the edge set of G can be partitioned into d edge disjoint complete matchings.

   Solution: For part (a), note that since G is bipartite we can count the edges of G by summing the degrees in either A or B. This gives

$$d|A| = \sum_{a \in A} d_G(a) = e(G) = \sum_{b \in B} d_G(b) = d|B|.$$

   As $d > 0$ this implies $|A| = |B|$.

   (b) Given $S \subset A$ let $\Gamma(S) := \{b \in B : ab \in E(G) \text{ for some } a \in S\} \subset B$. By Hall's theorem in order to prove G has a complete matching from A to B it is enough to show that $|\Gamma(S)| \geq |S|$ for all $S \subset A$. To see this, we will estimate edges between S and $\Gamma(S)$ in two ways; this technique is often referred to as 'double counting'. Note that

$$d|S| = \Big|\{(a, b) \in S \times \Gamma(S) : ab \in E(G)\}\Big| \leq d|\Gamma(S)|.$$

   The equality holds as each $a \in S$ has all $d_G(a) = d$ neighbours in $\Gamma(S)$ and the inequality holds since each $b \in \Gamma(S)$ has at most $d_G(b) = d$ neighbours in S. Dividing by d we see that the conditions of Hall's theorem hold for G.

   (c) By induction on d. If $d = 1$ then the edges of G form a complete matching. We will prove the result for $d \geq 2$ assuming by induction that it holds for smaller degree. From (b) there is a complete matching $\mathcal{M}$ in G. Let G' denote the graph obtained from G by deleting the edges of $\mathcal{M}$. All vertices in G' have degree $d - 1$ and so the edges of G' can be partitioned into $d - 1$ complete matching $\mathcal{M}_1, \ldots, \mathcal{M}_{d-1}$. Combined with $\mathcal{M}$ this gives the required partition.

2. Let $\mathcal{P}(n)$ denote the power set of $[n] := \{1, \ldots n\}$. For $A, B \in \mathcal{P}(n)$, we define the *symmetric difference of* A *and* B is $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

(a) Suppose $\mathcal{A} \subset \mathcal{P}(n)$, and there do not exist $A, B \in \mathcal{A}$ with $|A \triangle B| = 1$. How large can $|\mathcal{A}|$ be?

(b) For $n \geq 1$, give two examples of $\mathcal{A}$ with maximal size. Are there any others?

Solution: It may be helpful to note that if two sets have symmetric difference of size 1, then the corresponding vertices in the hypercube are adjacent. (See the introductory lecture for this correspondence.)

For (a), we define a bipartite graph as follows. Let $V = \mathcal{A}$ and let $W = \mathcal{P}(n) \setminus \mathcal{A}$, and join $A \in V$ and $B \in W$ if and only if their symmetric difference has size 1. Now note that every vertex in V has degree n, while every vertex in W has degree at most n. Double counting the set E of edges, we see that

$$|A|n = |E| \leq |B|n, \tag{1}$$

and so $|A| \leq |B|$. We deduce that $|A| \leq |\mathcal{P}(n)|/2 = 2^{n-1}$.

For (b), two examples are the sets of even size and the sets of odd size.

These are in fact the only examples. If $\mathcal{A}$ is an example with $|\mathcal{A}| = 2^{n-1}$, then we must have equality in (1). So every vertex in W has degree n. Now suppose $\mathcal{A}$ contains a set F of even size. If we add or delete an element to F then we get a set F' in $\mathcal{P}(n) \setminus \mathcal{A}$; and if we add or delete an element to F' then we get a set F'' in $\mathcal{A}$ again. You can get between any two sets of even size by changing two elements at a time, so $\mathcal{A}$ must contain all sets of even size (and so no sets of odd size). On the other hand, if $\mathcal{A}$ contains no sets of even size, then it must contain all sets of odd size.

3. Let $[n]^{(i)} := \big\{A \subset \{1, \ldots, n\} : |A| = i\big\}$ and suppose that $i \leq n/2$. Prove that there is a bijection $f : [n]^{(i)} \to [n]^{(i)}$ such that $A \cap f(A) = \emptyset$ for every A.

Solution: Let's set up a bipartite graph $G = (V, W)$. Let $V = W = [n]^{(i)}$, and join $A \in V$ and $B \in W$ if $A \cap B = \emptyset$. Since $i \leq n/2$, the graph G does have edges; and it is easy to see that every vertex has the same degree (in fact $\binom{n-i}{i}$). We know from Question 1 that this graph contains a complete matching: use this to define the mapping.

4. (a) Prove that $|\mathcal{P}(n)| = 2^n$.

(b) Suppose a set $A \in \mathcal{P}[n]$ is selected uniformly at random. Let X denote the random variable given by $X(A) := |A|$. Prove that $\mathbb{E}(X) = n/2$ and $\text{var}(X) = n/4$.

(c) Use Chebyshev's inequality and (b) to show that given $\epsilon > 0$ there is $C > 0$ such that at least $(1-\epsilon)2^n$ sets $A \subset [n]$ satisfy $\big||A| - \frac{n}{2}\big| \leq C\sqrt{n}$.

Solution: (a) The map which sends the vector $(x_1, \ldots, x_n) \in \{0,1\}^n$ to the set $\{i \in [n] : x_i = 1\} \in \mathcal{P}[n]$ is a bijection, and so $|\mathcal{P}[n]| = |\{0,1\}^n| = 2^n$.

To see (b), note that X can be written as a sum of indicator random variables $X = \sum_{i\in[n]} X_i$, where $X_i(A) = 1$ if $i \in A$ and $X_i(A) = 0$ if $i \notin A$. We have $\mathbb{E}(X_i) = \mathbb{P}(i \in A) = \frac{1}{2}$ for each $i \in [n]$ and so linearity of expectation gives

$$\mathbb{E}(X) = \mathbb{E}(\sum_{i\in[n]} X_i) = \sum_{i\in[n]} \mathbb{E}(X_i) = \frac{n}{2}.$$

To see the variance calculation recall that

$$\begin{aligned}\mathbb{V}ar(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 &= \mathbb{E}\Big((\sum_{i\in[n]} X_i)(\sum_{j\in[n]} X_j)\Big) - \frac{n^2}{4} \\ &= \sum_{i\in[n]} \mathbb{E}\Big(X_i^2\Big) + \sum_{i,j\in[n]:i\neq j} \mathbb{E}\Big(X_i X_j\Big) - \frac{n^2}{4} \\ &= n \cdot \frac{1}{2} + n(n-1) \cdot \frac{1}{4} - \frac{n^2}{4} = \frac{n}{4}.\end{aligned}$$

Here we used $\mathbb{E}(X_i^2) = \mathbb{E}(X_i) = \frac{1}{2}$ and $\mathbb{E}(X_i X_j) = \mathbb{P}(i, j \in A) = \frac{1}{4}$ for $i \neq j$.

For (c) note that for any $t > 0$ by Chebyshev's inequality we have

$$\mathbb{P}\Big(\big||A| - \frac{n}{2}\big| \geq t\Big) = \mathbb{P}\Big(\big|X - \mathbb{E}(X)\big| \geq t\Big) \leq \frac{\mathbb{V}ar(X)}{t^2} = \frac{n}{4t^2}.$$

If we set $t = Cn^{1/2}$ where $C = \epsilon^{-1/2}/2$ then gives $\mathbb{P}\big(\big||A| - \frac{n}{2}\big| \geq t\big) \leq \epsilon$. Since the sets were selected uniformly at random, this is equivalent to the statement that at most $\epsilon 2^n$ sets $A \in \mathcal{P}[n]$ satisfy $\big||A| - \frac{n}{2}\big| \geq t$.