STOCHASTIC DIFFERENTIAL EQUATIONS
MATH C8.1 - 2019 - SHEET 1

(i) Let $(B_t)_{t\in[0,\infty)}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove using the properties of Brownian motion that the following three processes are martingales.

(a) The process

$$(B_t)_{t\in[0,\infty)}.$$

(b) The process

$$\left(B_t^2 - t\right)_{t\in[0,\infty)}.$$

(c) And for every $\alpha \in \mathbb{R}$, the process

$$\left(\exp\left(\alpha B_t - \frac{\alpha^2 t}{2}\right)\right)_{t\in[0,\infty)}.$$

(ii) Let $(B_t)_{t\in[0,\infty)}$ be a standard Brownian motion on $\mathbb{R}^2$ and for $R \in (0,\infty)$ let B_R denote the ball of radius R centered at the origin. For every $t \in (0,\infty)$, compute

$$\mathbb{P}[B_t \in B_R],$$

and thereby prove that

(i) $\mathbb{P}(B_t \notin B_{\sqrt{2\lambda t}}) = e^{-\lambda}$,

(ii) and that, for the Lebesgue measure $|B_R|$ of B_R,

$$\lim_{R\to 0} \frac{\mathbb{P}[B_t \in B_R]}{|B_R|} = \frac{1}{2\pi t}.$$

What happens in dimension three?

(iii) Let $(B_t)_{t\in[0,\infty)}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(i) Prove that $W_t = tB_{\frac{1}{t}}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) For every $a \in [0,\infty)$, let T_a denote the stopping time

$$T_a = \inf\{t \in [0,\infty) \colon B_t = a\}.$$

Show using the reflection principle that

$$\mathbb{P}(T_a < t) = 2\mathbb{P}(B_t > a),$$

and thereby deduce that the probability density function f_a of T_a on $[0,\infty)$ is

$$f_a(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right).$$

(iii) For every $a \in [0,\infty)$, define the random time

$$S_a = \sup\{t \in [0,\infty) \colon B_t = at\}.$$

Is S_a a stopping time? Show that $S_a = {}^1/_{T_a}$ in distribution and find $\mathbb{E}[S_a]$ and $\mathbb{E}[B_{S_a}]$.

(iv) A real-valued centered Gaussian process is a real valued process $(X_t)_{t\in[0,\infty)}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite dimensional distributions that are normally distributed and mean zero. A centered Gaussian process $(X_t)_{t\in[0,\infty)}$ is a fractional Brownian motion with Hurst parameter $h \in (0,1)$ if $\mathbb{P}[X_0 = 0] = 1$ and if, for every $s, t \in [0,\infty)$,

$$\mathbb{E}[X_t X_s] = \frac{1}{2}\left(t^{2h} + s^{2h} - |t-s|^{2h}\right).$$

(a) Show that, for every $s \leq t \in [0,\infty)$, the increment $X_t - X_s$ has mean zero and variance $|t-s|^{2h}$.

(b) Show that for every $p \in (0,\infty)$ there exists $c_p \in (0,\infty)$ such that, for every $s \leq t \in [0,\infty)$,

$$\mathbb{E}[|X_t - X_s|^p] = c_p\,|t-s|^{hp}.$$

(c) Show that fractional Brownian motion has a continuous modifcation, and determine its Hölder exponent. (Hint: Use Kolmogorov's continuity criterion.)

(d) For each $h \in (0,1)$, determine the value $p \in (0,\infty)$ so that fractional Brownian motion with Hurst parameter h has finite and nonzero p-variation. Is fractional Brownian motion a semimartingale for any $h \in (0,1)$?

(v) Let $(M_t)_{t\in[0,\infty)}$ be a continuous local martingale with $M_0 = 0$.

(a) Prove that if $\mathbb{E}[\langle M\rangle_\infty] < \infty$ then M is L^2-bounded in the sense that

$$\sup_{t\in[0,\infty)} \mathbb{E}[M_t^2] < \infty.$$

In this case, conclude that the process $\left(M_t^2 - \langle M\rangle_t\right)_{t\in[0,\infty)}$ is uniformly integrable.

(b) Prove that M converges almost surely as $t \to \infty$ on the set $\{\langle M\rangle_\infty < \infty\}$.

(vi) Suppose that $(M_t)_{t\in[0,\infty)}$ is a bounded continuous martingale with finite variation. Prove that

$$M_t^2 = M_0^2 + 2\int_0^t M_s\,\mathrm{d}M_s,$$

where the final integral is almost surely well-defined. (Hint: Cite here known results about one-dimensional functions of bounded variation.) Deduce that M is almost surely constant.