STOCHASTIC DIFFERENTIAL EQUATIONS
MATH C8.1 - 2019 - SHEET 2

(i) We say that a function $u\colon \mathbb{R}^2 \to \mathbb{R}$ is C^2-bounded if u is twice-differentiable and satisfies

$$\sup_{x\in\mathbb{R}^2} \left(|u(x)| + |\nabla u(x)| + \left|\nabla^2 u(x)\right|\right) < \infty.$$

Prove that every C^2-bounded function u satisfying

$$\Delta u = \frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0,$$

is constant. (Hint: Apply Itô's formula to the composition $(u(B_t))_{t\in[0,\infty)}$, for $(B_t)_{t\in[0,\infty)}$ a standard two-dimensional Brownian motion.) Conclude that every bounded holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ is constant.

(ii) Let $(M_t)_{t\in[0,\infty)}$ be a continuous local martingale vanishing at zero.

(a) Show that the intervals of constancy of the maps $t \mapsto M_t$ and $t \mapsto \langle M\rangle_t$ coincide almost surely.

(b) Show that if, for every $\xi \in \mathbb{R}$, for every $s \le t \in [0,\infty)$,

$$\mathbb{E}[\exp(i\xi(M_t - M_s))|\mathcal{F}_s] = \exp\left(-\frac{\xi^2(t-s)}{2}\right),$$

then $(M_t)_{t\in[0,\infty)}$ is a Brownian motion.

(iii) Let $(B_t = (B_t^1, \ldots, B_t^d))_{t\in[0,\infty)}$ be a standard d-dimensional Brownian motion. Let $(F_t = (F_t^1, \ldots, F_t^d))_{t\in[0,\infty)}$ be a continuous, adapted, d-dimensional stochastic process that satisfies, for every $i \in \{1,\ldots,d\}$, for every $t \in (0,\infty)$,

$$\mathbb{E}[\int_0^t \left|F_s^i\right|^2 \,\mathrm{d}s] < \infty.$$

(a) Prove that, for every $i, j \in \{1,\ldots,d\}$, for every $t \in (0,\infty)$,

$$\langle B^i, B^j\rangle_t = \delta_{ij} t = \begin{cases} t & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(b) Prove that, for every $i, j \in \{1,\ldots,d\}$, for every $t \in (0,\infty)$,

$$\langle \int_0^\cdot F_s^i \,\mathrm{d}B_s^i, \int_0^\cdot F_s^j \,\mathrm{d}B_s^j\rangle_t = \delta_{ij} \int_0^t F_s^i F_s^j \,\mathrm{d}s.$$

(c) Prove that the process $(X_t)_{t\in[0,\infty)}$ defined by

$$X_t = \left(\sum_{i=1}^d \int_0^t F_s^i \,\mathrm{d}B_s^i\right)^2 - \sum_{i=1}^d \int_0^t \left(F_s^i\right)^2 \,\mathrm{d}s,$$

is a martingale.

(d) Prove that, for every $\lambda, t \in (0,\infty)$,

$$\mathbb{P}\left[\left(\sup_{s\in[0,t]} \left|\sum_{i=1}^d \int_0^s F_r^i \,\mathrm{d}B_r^i\right|\right) \ge \lambda\right] \le \lambda^{-2} \sum_{i=1}^d \int_0^t \mathbb{E}[(F_t^i)^2] \,\mathrm{d}s.$$

(iv) Let $(B_t)_{t\in[0,\infty)}$ be a standard Brownian motion on a filtered probability space

$$(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,\infty)}, \mathbb{P}).$$

Let X be a finite $\mathcal{G}_0$-measurable positive random variable that is independent of the Brownian motion. Let $(M_t = B_{tX})_{t\in[0,\infty)}$ and define the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ by

$$\mathcal{F}_t = \sigma(B_{sX} \colon s \in [0,t]).$$

(a) Show that M is a local martingale with respect to $(\mathcal{F}_t)_{t\in[0,\infty)}$.
(b) Show that M is a martingale if and only if $\mathbb{E}[\sqrt{X}] < \infty$.
(c) Calculate $(\langle M\rangle_t)_{t\in[0,\infty)}$.
(d) Let $(A_t)_{t\in[0,\infty)}$ be an increasing process vanishing at zero that is independent of $(B_t)_{t\in[0,\infty)}$. Define the filtration $(\mathcal{F}^A_t)_{t\in[0,\infty)}$ by

$$F^A_t = \sigma(B_{A_s} \colon s \in [0,t]).$$

Show that $(B_{A_t})_{t\in[0,\infty)}$ is a local $\mathcal{F}^A_t$-martingale, find conditions that guarantee that $(B_{A_t})_{t\in[0,\infty)}$ is a $\mathcal{F}^A_t$-martingale, and compute its quadratic variation process.

(v) Let $(B_t)_{t\in[0,\infty)}$, $(W_t)_{t\in[0,\infty)}$ be two independent standard Brownian motions. Find the stochastic differential equations satisfied by the following processes $(X_t)_{t\in[0,\infty)}$, and determine which are martingales.
(a) $X_t = \exp(\frac{t}{2})\cos(B_t)$
(b) $X_t = tB_t$
(c) $X_t = (B_t + t)\exp(-B_t - \frac{t}{2})$
(d) $X_t = (B_t)^2 + (W_t)^2$

(vi) Let $(B_t)_{t\in[0,\infty)}$ be a standard d-dimensional Brownian motion with $B_0 \neq 0$, for $d \geq 2$. Let $(X_t)_{t\in[0,\infty)}$ be the process

$$X_t = \|B_t\| = \sqrt{(B^1_t)^2 + \ldots + (B^d_t)^2}.$$

(a) Find the SDE satisfied by $(X_t)_{t\in[0,\infty)}$ and show that

$$X_t = X_0 + \int_0^t \frac{d-1}{2X_s}\,\mathrm{d}s + W_t,$$

where $(W_t)_{t\in[0,\infty)}$ is standard one-dimensional Brownian motion.
(b) Let $\beta_k(t) = \mathbb{E}[|X_t|^{2k}]$ for every $k \in \mathbb{N}_0$ and $t \in (0,\infty)$. Prove that

$$\beta_k(t) = k(2(k-1) + d)\int_0^t \beta_{k-1}(s)\,\mathrm{d}s.$$

(c) Calculate the time $t \in [0,\infty)$ for which $\mathbb{E}[\|B_t\|^4] = \mathbb{E}[\|B_t\|^6]$.

(vii) Let $(B_t)_{t\in[0,\infty)}$ be a standard one-dimensional Brownian motion. Prove that, for every $x \in \mathbb{R}$,

$$X^x_t = \int_0^t \operatorname{sgn}(B_s - x)\,\mathrm{d}B_s,$$

is a Brownian motion where

$$\operatorname{sgn}(y) = \begin{cases} 1 & \text{if } y \geq 0, \\ -1 & \text{if } y < 0. \end{cases}$$

(viii) Let $(B_t^1, B_t^2)_{t\in[0,\infty)}$ be a standard two-dimensional Brownian motion. Prove that the process $((X_t^1, X_t^2))_{t\in[0,\infty)}$ defined by

$$X_t^1 = \int_0^t \cos(B_s^1)\,\mathrm{d}B_s^1 - \int_0^t \sin(B_s^1)\,\mathrm{d}B_s^2,$$
$$X_t^2 = \int_0^t \sin(B_s^1)\,\mathrm{d}B_s^1 + \int_0^t \cos(B_s^1)\,\mathrm{d}B_s^2,$$

is a standard two-dimensional Brownian motion.

(ix) Let $(X_t)_{t\in[0,\infty)}$ and $(Y_t)_{t\in[0,\infty)}$ be continuous semimartingales. Define the stochastic exponential $(\mathcal{E}(X)_t)_{t\in[0,\infty)}$ to be the process

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{\langle X\rangle_t}{2}\right).$$

Prove that there exists a unique continuous semimartingale $(Z_t)_{t\in[0,\infty)}$ such that

$$Z_t = Y_t + \int_0^t Z_s\,\mathrm{d}X_s,$$

and that

$$Z_t = \mathcal{E}(X)_t\left(Y_0 + \int_0^t \mathcal{E}(X)_s^{-1}\,\mathrm{d}Y_s - \int_0^t \mathcal{E}(X)_s^{-1}\,\mathrm{d}\langle X, Y\rangle_s\right).$$