STOCHASTIC DIFFERENTIAL EQUATIONS
MATH C8.1 - 2019 - SHEET 3

1. Sheet 3

(i) For every probability measure μ on $\mathbb{R}^d$ let $\hat{\mu}$ denote the Fourier transform defined for every $\xi \in \mathbb{R}^d$ by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} \exp\left(i\langle x, \xi\rangle\right) \mu(\,\mathrm{d}x).$$

In particular, if X is an $\mathbb{R}^d$-valued random variable with distribution μ_X, then the Fourier transform is the characteristic function of X in the sense that

$$\mathbb{E}\left[\exp(i\langle X, \xi\rangle)\right] = \int_{\mathbb{R}^d} \exp(i\langle x, \xi\rangle)\mu_X(\,\mathrm{d}x) = \hat{\mu}_X(\xi).$$

For probability measures μ and ν on $\mathbb{R}^d$, prove that $\mu = \nu$ if and only if $\hat{\mu} = \hat{\nu}$. Hint: For a Schwarz function ϕ, compute

$$\int_{\mathbb{R}^d} \phi(\xi)\hat{\mu}(\xi)\,\mathrm{d}\xi.$$

Show that if X is a normally distributed random variable with mean zero and variance $t \in (0, \infty)$ then

$$\hat{\mu}_X(\xi) = \exp\left(-\frac{\xi^2 t}{2}\right).$$

(ii) Let $(B_t)_{t\in[0,\infty)}$ be a standard $\mathcal{F}_t$-Brownian motion. Let $(M_t)_{t\in[0,\infty)}$ be an L^2-bounded $\mathcal{F}_t$-martingale in the sense that

$$\sup_{t\in[0,\infty)} \mathbb{E}\left[M_t^2\right] < \infty.$$

Prove that there exists a unique predictable process $(H_t)_{t\in[0,\infty)} \in L^2(B)$ such that, for every $t \in [0, \infty)$,

$$M_t = \mathbb{E}[M_0] + \int_0^t H_s\,\mathrm{d}B_s.$$

(iii) Let $(B_t)_{t\in[0,\infty)}$ be a standard one-dimensional Brownian motion. Let $\mathrm{C}([0,\infty);\mathbb{R})$ denote the space of continuous paths from $[0,\infty)$ into $\mathbb{R}$. Let $\mu, \sigma\colon [0,\infty) \times \mathrm{C}([0,\infty),\mathbb{R}) \to R$ be bounded functions in the sense that there exists $K_1 \in (0,\infty)$ such that, for every $t \in [0,\infty)$ and continuous path $(X_t)_{t\in[0,\infty)}$,

$$(|\mu(t, X_\cdot)| + |\sigma(t, X_\cdot)|) \leq K_1,$$

and which are Lipschitz continuous in the sense that there exists $K_2 \in (0,\infty)$ such that, for every $t \in [0,\infty)$, for every pair of continuous paths $(X_t)_{t\in[0,\infty)}$ and $(Y_t)_{t\in[0,\infty)}$,

$$(|\sigma(t, X_\cdot) - \sigma(t, Y_\cdot)| + |\mu(t, X_\cdot) - \mu(t, Y_\cdot)|) \leq K_2 \sup_{s\in[0,t]} |X_s - Y_s|\,.$$

Prove that there exists a jointly continuous process $(X_t^x)_{x\in\mathbb{R}^d,t\in[0,\infty)}$ such that, for every $x\in\mathbb{R}^d$ and $t\in[0,\infty)$,

$$X_t^x = x + \int_0^t \mu(s, X_\cdot^x)\,\mathrm{d}s + \int_0^t \sigma(s, X_\cdot^x)\,\mathrm{d}B_s \text{ almost surely.} \tag{1.1}$$

Hint: The issue is proving continuity in space. In class, we proved that for every $x\in\mathbb{R}^d$ there exists a continuous in time solution $(X_t^x)_{t\in[0,\infty)}$ of (1.1). For every $p\in[2,\infty)$ and $t\in(0,\infty)$, use the inequality $|a+b+c|^p\le 3^{p-1}\left(|a|^p+|b|^p+|c|^p\right)$ to prove that, for each $x,y\in\mathbb{R}^d$,

$$\begin{aligned}&\sup_{s\in[0,t]}|X_s^x-X_s^y|^p\\ &\le 3^{p-1}|x-y|^p\\ &\quad+3^{p-1}\left(\sup_{s\in[0,t]}\left|\int_0^s \mu(r,X_r^x)-\mu(r,X_r^y)\,\mathrm{d}r\right|^p+\sup_{s\in[0,t]}\left|\int_0^s \sigma(r,X_r^x)-\sigma(r,X_r^y)\,\mathrm{d}B_r\right|^p\right).\end{aligned}$$

Then prove using the Burkholder-Davis-Gundy inequality, Hölder's inequality, and $p\in[2,\infty)$ that there exists $C_p\in(0,\infty)$ such that

$$\mathbb{E}\left[\sup_{s\in[0,t]}\left|\int_0^s \sigma(r,X_r^x)-\sigma(r,X_r^y)\,\mathrm{d}B_r\right|^p\right]\le C_pK_2^pt^{\frac{p-2}{p}}\int_0^t\mathbb{E}\left[\sup_{s\in[0,r]}|X_s^x-X_s^y|^p\right]\mathrm{d}r.$$

Deduce using Jensen's inequality and $p\in[2,\infty)$ that

$$\sup_{s\in[0,t]}\left|\int_0^s \mu(r,X_r^x)-\mu(r,X_r^y)\,\mathrm{d}r\right|^p\le K_2^pt^{p-1}\int_0^t\sup_{s\in[0,r]}|X_s^x-X_s^y|^p\,\mathrm{d}r.$$

Conclude that, for every $t\in[0,\infty)$,

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\in[0,t]}|X_s^x-X_s^y|^p\right]\\ &\le 3^{p-1}\left(|x-y|^p+\left(C_pK_2^pt^{\frac{p-2}{p}}+K_2^pt^{p-1}\right)\int_0^t\mathbb{E}\left[\sup_{s\in[0,r]}|X_s^x-X_s^y|^p\right]\mathrm{d}r\right).\end{aligned}$$

Prove using the Gronwall inequality that there exists a constant $c(t,p)\in(0,\infty)$ depending on $t\in[0,\infty)$ and $p\in[2,\infty)$ such that

$$\mathbb{E}\left[\sup_{s\in[0,t]}|X_s^x-X_s^y|^p\right]\le c(t,p)\,|x-y|^p.$$

Deduce using the Komogorov continuity criterion that there exists a bicontinuous modification of the process $(X_t^x)_{x\in\mathbb{R}^d,t\in[0,\infty)}$ which solves (1.1).

(iv) Let $(W_t^1,W_t^2,W_t^3)_{t\in[0,\infty)}$ be a three-dimensional Brownian motion, and assume that W_0 takes values in $\mathbb{R}^d\setminus\{0\}$ and that W_0 is independent of $(W_t-W_0)_{t\in[0,\infty)}$. Define the Euclidean norm

$$|W|=\left((W_t^1)^2+(W_t^2)^2+(W_t^3)^2\right)^{\frac{1}{2}}.$$

(a) Show that $(|W_t|^{-1})_{t\in[0,\infty)}$ is a local martingale. Hint: If $d\ge 3$, the function $|x|^{2-d}$ is harmonic on $\mathbb{R}^d\setminus\{0\}$.

(b) Suppose that $W_0=y\in\mathbb{R}^d$ and for every $t\in[0,\infty)$ define $M_t=|W_{1+t}-y|^{-1}$. Prove by direct calculation that $\mathbb{E}[M_t^2]={}^1/_{1+t}$. Deduce that $(M_t)_{t\in[0,\infty)}$ is L^2-bounded and hence uniformly integrable.

(c) Show that $(M_t)_{t\in[0,\infty)}$ is a local martingale and a supermartingale.

(d) Use the martingale convergence theorem to prove that $(M_t)_{t\in[0,\infty)}$ is not a martingale.

(v) Let $(B_t)_{t\in[0,\infty)}$ be a standard one-dimensional Brownian motion. Prove that

$$B_t^4 = 3t^2 + \int_0^t \left(12(t-s)B_s + 4B_s^3\right) \mathrm{d}B_s.$$

(vi) Let $d_1, d_2 \in \mathbb{N}$. Let $(B_t)_{t\in[0,\infty)}$ be a standard d_2-dimensional Brownian motion. Let μ be a constant $(d_1 \times d_1)$-matrix and let σ be a constant $(d_1 \times d_2)$-matrix.

(a) For every $x \in \mathbb{R}^d$, find the unique strong solution $(X_t^x, B_t)_{t\in[0,\infty)}$ to the equation

$$\begin{cases} \mathrm{d}X_t^x = \mu X_t^x \,\mathrm{d}t + \sigma \,\mathrm{d}B_t & \text{in } (0,\infty), \\ X_0^x = x. \end{cases}$$

Hint: For a $d_1 \times d_1$-matrix A, use properties of the matrix exponential

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.$$

The solution itself will be expressed in terms of a stochastic integral.

(b) Find the distribution of X_t^x for every $t \in [0,\infty)$.

(c) Let $d_1 = d_2 = 1$. Prove that, for every bounded measurable function $f\colon \mathbb{R}^d \to \mathbb{R}$,

$$\mathbb{E}\left[f(X_t^x)\right] = \mathbb{E}'\left[f\left(x\exp(\mu t) + N\sqrt{\frac{\sigma^2}{2\mu}\left(\exp(2t\mu) - 1\right)}\right)\right],$$

where $\mathbb{E}'$ denotes the expectation on any probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ carrying a normally distributed random variable N with mean zero and variance one.

(vii) Let $\lambda, \nu, \sigma \in (0,\infty)$ and let $\mu\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$\mu(x) = \lambda(\nu - x).$$

Let $(B_t)_{t\in[0,\infty)}$ be a standard one-dimensional Brownian motion.

(a) For every $x \in \mathbb{R}$, find the unique strong solution $(X_t^x, B_t)_{t\in[0,\infty)}$ to the equation

$$\begin{cases} \mathrm{d}X_t^x = \mu(X_t^x) \,\mathrm{d}t + \sigma \,\mathrm{d}B_t & \text{in } (0,\infty), \\ X_0^x = x. \end{cases}$$

(b) Calculate the mean and variance of X_t^x, for each $x \in \mathbb{R}^d$ and $t \in [0,\infty)$.

(c) Let $\nu = 1$. Show that $(Y_t^x = (X_t^x)^2, B_t)_{t\in[0,\infty)}$ is a strong solution to the equation

$$\begin{cases} \mathrm{d}Y_t^x = \left(-2\lambda Y_t + \sigma^2\right) \,\mathrm{d}t + 2\sigma\sqrt{Y_t}\,\mathrm{d}B_t & \text{in } (0,\infty), \\ Y_0^x = x^2. \end{cases}$$

(viii) Let $\mu, \sigma \in \mathbb{R}$. Let $(B_t)_{t\in[0,\infty)}$ be a standard one-dimensional Brownian motion. For every $x \in \mathbb{R}$, use the stochastic exponential to find a strong solution $(X_t^x, B_t)_{t\in[0,\infty)}$ to the equation

$$\begin{cases} \mathrm{d}X_t^x = \mu \,\mathrm{d}t + \sigma X_t^x \,\mathrm{d}B_t & \text{in } (0,\infty), \\ X_0^x = x. \end{cases}$$

(ix) Let $(M_t)_{t\in[0,\infty)}$ be a continuous local martingale vanishing at zero, and let $(L_t^0)_{t\in[0,\infty)}$ denote the local time of $(M_t)_{t\in[0,\infty)}$ at zero.

(a) Prove that

$$\inf\{t \in [0,\infty) \colon L_t^0 > 0\} = \inf\{t \in [0,\infty) \colon \langle M \rangle_t > 0\} \text{ almost surely.}$$

(b) Prove that if $\alpha \in (0,1)$ and $(M_t)_{t\in[0,\infty)}$ is not identically zero, then $(|M_t|^\alpha)_{t\in[0,\infty)}$ is not a semimartingale.