Problem sheet 1

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All questions use natural units, where Newton's constant G and the speed of light c are both equal to 1.

Questions marked with a star * are optional extension questions which go beyond the scope of the course. They will not be discussed in class unless all other questions have already been covered. You are advised to only attempt these questions if you have already completed the other questions on the sheet.

Questions/comments/correction to Joseph.Keir@maths.ox.ac.uk

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1. *(Carrollian spacetimes)*

An apparently natural choice of a spacetime structure is given by *Carrollian spacetime*, in which spacetime is given by $\mathbb{E}^4$ together with a *degenerate* inner product: for X, $Y \in \mathbb{R}^4$,

$$\langle X, Y \rangle = g(X, Y) \geq 0.$$

We are also provided by a preferred 'timelike' vector T, with the property that

$$g(T, X) = 0 \quad \text{for all vectors } X \in \mathbb{R}^4.$$

Moreover, if $g(Y, X) = 0$ for all vectors X, then $Y \propto T$ – in other words, g has rank 3.

The idea is that g gives us a way of measuring distances in space, while T gives us a way of measuring times. Suppose an observer uses coordinates (t, x^i), with respect to which $T = (1, 0)$, and

$$g = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \delta_{ij} \end{array} \right).$$

If p, $q \in \mathbb{E}^4$ then

- the spatial distance from p to q is $\sqrt{g(q-p, q-p)}$
- writing $(q - p)$ in coordinates as $\big((q-p)^0, (q-p)^i\big)$, the time from p to q is $(q-p)^0$.

a) In a Carrollian spacetime, a second observer might choose alternative coordinates $(t', (x')^i)$. Assume that these are related to the original coordinates by a linear map. Then, if the forms of both the vector field T and the metric g are invariant, show that

$$t' = t + b_i x^i$$
$$(x')^i = R_j{}^i x^j,$$

where $R \in O(3)$ and b_i are constants. More generally, if we also allow changes of origin, the *Carrollian transformations* are given by

$$t' = t + b_i x^i + t_0$$
$$(x')^i = R_j{}^i x^j + x_0^i$$

for constants t_0 and x_0^i.

b) Hence show that this second observer will measure the time between events p and q to be

$$(q-p)^0 + b_i(q-p)^i,$$

but they will measure the same distance between the two events as the first observer. So, in a Carrollian spacetime, all observers will agree on the distance between two events, but not the time!

c) Writing

$$\begin{aligned}
\partial_t &= \tfrac{\partial}{\partial t}\big|_{x,y,z} & \partial'_{t'} &= \tfrac{\partial}{\partial t'}\big|_{x',y',z'} \\
\partial_x &= \tfrac{\partial}{\partial x}\big|_{t,y,z} & \partial'_{x'} &= \tfrac{\partial}{\partial x'}\big|_{t',y',z'} \\
\partial_y &= \tfrac{\partial}{\partial y}\big|_{t,x,z} & \partial'_{y'} &= \tfrac{\partial}{\partial y'}\big|_{t',x',z'} \\
\partial_z &= \tfrac{\partial}{\partial z}\big|_{t,x,y} & \partial'_{z'} &= \tfrac{\partial}{\partial z'}\big|_{t',x',y'},
\end{aligned}$$

i) show that

$$\begin{aligned}
\partial_t &= \partial'_{t'} \\
\partial_i &= b_i \partial'_{t'} + R_i{}^j \partial'_j
\end{aligned}$$

and therefore that the only first order differential operators which are invariant under Carrollian transformations are proportional to time derivatives.

ii) Show also that the only second order differential operators which are invariant under Carrollian transformations are proportional to ∂_t^2.

iii) Repeat parts **i)** and **ii)**, replacing Carrollian transformations with Gallilean transformations

$$\begin{aligned}
t' &= t + t_0 \\
(x')^i &= R_j{}^i x^j + v^i t + x_0^i
\end{aligned}$$

and showing that, this time, the Laplacian is invariant.

Dynamics in Carrollian spacetimes are sometimes called *ultralocal*: physical laws which are consistent with the symmetry only involve derivatives in the time direction, i.e. the direction of the vector T. In other words, each point in space evolves separately, in isolation from the other points in space! Both Galilean spacetime and Carrollian spacetimes can be thought of as particular limits of Minkowski space: in the Galilean case the speed of light c is taken to infinity, opening up the light cones, while in the Carrollian case c is taken to zero, closing the light cones up.

> "Well, in our country," said Alice, still panting a little, "youd generally get to somewhere else if you run very fast for a long time, as weve been doing."
>
> "A slow sort of country!" said the Queen. "Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!"
>
> – Lewis Carroll, *Through the looking glass*

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2. *(Newtonian tidal forces)*

In this question we work in the framework of Newtonian gravity.

Suppose a particle of mass m is situated at some large radius r_0 above the centre of the Earth, which has mass M. Initially this particle is at rest.

a) At time $t > 0$ the particle is at a radius $r > 0$ above the centre of the Earth.

i) Show that

$$t = \int_r^{r_0} \left(\frac{2M}{r'} - \frac{2M}{r_0} \right)^{-\frac{1}{2}} \mathrm{d}r'.$$

ii) Hence show that, if $r_0 - r \ll r_0$ (which will be true for sufficiently small times) then

$$r = r_0 - \frac{M}{2r_0^2} t^2 + \mathcal{O}(t^3).$$

b) Now suppose that there are two such particles, both initially placed at a radius r_0 above the centre of the earth, separated by a 'horizontal' distance x_0. Let the distance between the two particles at time t be x. Show that

$$x = x_0 \left(1 - \frac{M}{2r_0^3} t^2 \right) + \mathcal{O}(t^3),$$

and so

$$\ddot{x} = -\frac{M}{r^3} x + \mathcal{O}(t).$$

So, if one particle represents the path of an observer, then this observer will see a 'force' acting on the other particle of magnitude $\frac{Mm}{r^3}x$, directed towards them. This is called a *tidal force*. Of course, it is not really a force, but a manifestation of the fact that the gravitational field is non-uniform. Note also that the 'force' is proportional to the distance between the particles x: point particles experience no tidal forces, while extended objects do.

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3. *(A circular version of the twin 'paradox')*

In this question we work in the framework of special relativity.

Alice and Bob are twins. Bob remains on Earth (which can be considered stationary) while Alice takes a ride on a spaceship. From Bob's point of view (i.e. using inertial coordinates in which Bob remains $x^i = 0$), Alice takes a circular path, beginning and ending at the Earth.

Suppose that when Alice returns to the Earth, from Bob's point of view the amount of time that has passed is T. Suppose also that the radius of the circle, as measured by Bob, is $v\frac{T}{2\pi}$, with $0 < v < 1$.

Show that, from Alice's point of view, the journey takes a time T_A, where

$$T_A = T\sqrt{1 - v^2}.$$

Thus, if the time T corresponds to several years, but v is close to 1 (so Alice travels at a large relative velocity) then Bob will end up several years older than Alice.

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4. *(Gravitational energy extraction)*

Bob has the following plan to extract energy from the Earth's gravitational field: from the top of a tower of height H, he will drop a ball with rest mass m. When this ball reaches the bottom of the tower, Bob has installed a machine that converts the ball into photons. These photons are then directed back to the top of the tower, where another machine captures the photons and reforms the original ball.

a) On the scale of this experiment, the gravitational field is approximately uniform. Assuming that the ball falls with a constant acceleration g, and that $gH \ll 1$ (so that terms of order (gH^2) can be neglected), show that the total energy of the ball when it reaches the bottom of the tower is

$$E = m + mgH.$$

b) The energy of a photon of frequency ν is $h\nu$, where h is Planck's constant. Using Maxwell's equations in vacuum

$$\begin{aligned}\operatorname{div}\mathbf{E} &= 0\\ \operatorname{curl}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0\\ \operatorname{div}\mathbf{B} &= 0\\ \operatorname{curl}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= 0,\end{aligned}$$

explain why Bob expects the energy of the photons to remain constant as they travel to the top of the tower.

c) Hence show that, assuming that the machines for converting balls into photons and vice versa are perfectly efficient, the excess energy returned to the top of the tower - which Bob plans to extract - is given by

$$mgH + \mathcal{O}((gH)^2).$$

d) Alice does not think Bob's plan will work. She proposes that the energy will remain constant throughout the process, and so the frequency of the photons must decrease as they gain height. Show that, if Alice is correct, then the frequency of the photons behaves as

$$\nu = \nu_0 \left(1 - gH + \mathcal{O}((gH)^2)\right),$$

where ν_0 is the frequency of the emitted photons. This would mean that there is a *redshift* of photons as they climb up the gravitational potential.

e) Using the equivalence principle, make an argument for who is correct out of Alice and Bob.

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5. *(Energy momentum tensors)*

a) A Klein-Gordon field ϕ satisfies the equation

$$\Box\phi - \mu^2\phi = 0,$$

where $\mu > 0$ is a constant called the *mass* of the Klein-Gordon field. The energy momentum tensor for this field is

$$T_{ab} = (\partial_a\phi)(\partial_b\phi) - \frac{1}{2}m_{ab}\Big((m^{-1})^{cd}(\partial_c\phi)(\partial_d\phi) + \mu^2\phi^2\Big).$$

Show that $\partial^a T_{ab} = 0$.

b) Maxwell's equations can be written in a covariant way as

$$\begin{aligned}\partial^a F_{ab} &= 0\\ \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} &= 0,\end{aligned}$$

where $F_{ab} = -F_{ba}$ is an antisymmetric tensor called the *electromagnetic field tensor* (see example sheet 0).

The associated energy momentum tensor is

$$T_{ab} = F_a{}^c F_{bc} - \frac{1}{4}m_{ab}F_{cd}F^{cd}.$$

Show that $\partial^a T_{ab} = 0$.

c) The energy momentum tensor for a perfect fluid is

$$T_{ab} = (\rho + p)u_a u_b + p m_{ab},$$

where ρ is the energy density of the fluid in its rest frame, p is the fluid pressure and u^a is the fluid 4-velocity. Since the fluid moves along timelike curves, we also require that

$$u_a u^a = -1$$

i) Show that the equation for the energy momentum tensor $\partial_a T^{ab} = 0$ implies the following two equations, which are the *relativistic Euler equations*

$$\begin{aligned} \partial_a(\rho u^a) + p\partial_a u^a = 0 \\ (\rho + p)u^b \partial_b u^a + \left((m^{-1})^{ab} + u^a u^b\right) \partial_b p = 0. \end{aligned}$$

(*Hint:* given a vector X^a, the projection of of X onto the plane orthogonal to u is given by $X^a + (u_b X^b)u^a$.)

To see how these equations are related to the standard (non-relativistic) Euler equations, set

$$u = \begin{pmatrix} 1 \\ \boldsymbol{u} \end{pmatrix},$$

and

$$\begin{aligned} |\boldsymbol{u}| &= \mathcal{O}(\epsilon) \\ \rho &= \mathcal{O}(1) \\ p &= \mathcal{O}(\epsilon^2). \end{aligned}$$

Also, time derivatives add an extra factor of ϵ, while spatial derivatives do not change the order of the quantity, so, for example,

$$\begin{aligned} \partial_t \rho &= \mathcal{O}(\epsilon) \\ \nabla \boldsymbol{u} &= \mathcal{O}(\epsilon) \end{aligned}$$

(where ∇ is the spatial gradient operator.) In all of these expressions, ϵ is considered to be very small.

ii) Explain the physical meaning of these conditions. (*Hint:* you may wish to reinstate factors of c.)

iii) Show that, for small ϵ, the leading order components of the relativistic Euler equations are

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \rho(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p, \end{aligned}$$

which is the standard form of the non-relativistic Euler equations.

***6.** *(The connection on Galilean spacetimes)*

Recall that Galilean spacetime $\mathbb{G}$ is modelled as a fibre bundle of $\mathbb{E}^3$ over $\mathbb{E}$. This means that:

1. There is a (continuous, surjective) map π, called the *projection map*,
$$\pi : \mathbb{G} \to \mathbb{E}.$$
2. For every point $p \in \mathbb{E}$, there is a small open neighbourhood U, with $p \in U \subset \mathbb{E}^3$ and a *local trivialization* (U, ϕ_U), which is a continuous bijection
$$\phi_U : \pi^{-1}(U) \to U \times \mathbb{E}^3.$$
3. The local trivialization should commute with the projection map, so that for any $p \in \mathbb{G}$, we have
$$\pi(p) = \tilde{\pi}_U(\phi_U(p)),$$
where $\tilde{\pi}_U$ is the canonical projection
$$\begin{aligned} \tilde{\pi}_U : \mathbb{E} \times \mathbb{E}^3 &\to \mathbb{E} \\ (x, y) &\mapsto x. \end{aligned}$$

These local trivializations are not to be thought of as in any way canonical or having a physical meaning: in fact, given one local trivialization it is easy to construct infinitely many other local trivializations covering the same region. Instead, they should be thought of like coordinates: very useful for doing calculations, but ultimately lacking any physical meaning.

Let $\gamma : [-1, 1] \to \mathbb{G}$ be a curve in the Galilean spacetime, with $\gamma(0) = p$.

a) Suppose that $p \in \pi^{-1}(U)$. With respect to some local trivialization (U, ϕ_U), and close to the point p, we can define a corresponding curve in $\mathbb{E} \times \mathbb{E}^3$ as follows:

$$\tilde{\gamma}_U := \phi_U \circ \gamma.$$

Show that the tangent vector to $\tilde{\gamma}_U$ at p, defined as

$$\dot{\tilde{\gamma}}_U := \frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{\gamma}_U(\tau)\big|_{\tau=0},$$

takes a value in the vector space $\mathbb{R} \times \mathbb{R}^3$, *not* the affine space $\mathbb{E} \times \mathbb{E}^3$. (*Hint*: consider the definition of the derivative as a limit.)

b) Suppose that $f : \mathbb{E} \times \mathbb{E}^3 \to \mathbb{E} \times \mathbb{E}^3$ is a smooth map. Define the differential $\mathrm{d}f\big|_q$ as follows: for any curve γ through $q \in \mathbb{E} \times \mathbb{E}^3$, with $\gamma(0) = q$ and with tangent vector at q $\dot{\gamma}$,

$$\mathrm{d}f\big|_q(\dot{\gamma}) = \frac{\mathrm{d}}{\mathrm{d}\tau}f(\gamma(\tau))\big|_{\tau=0}.$$

Show that the differential $\mathrm{d}f\big|_q$ can be used to define a linear map (one that depends on the trivialization!) from $\mathbb{R} \times \mathbb{R}^3$ to $\mathbb{R}$. Note again that this is a linear map between vector spaces, not affine spaces. (*Hint*: define $\tilde{f}_U := f \circ \phi_U^{-1}$.)

c) Now suppose that there is a different local trivialization (V, ϕ_V), with $p \in \pi^{-1}(V)$. We can define a map

$$\begin{aligned} \psi_{U,V} : (U \cap V) \times \mathbb{E}^3 &\to (U \cap V) \times \mathbb{E}^3 \\ q &\mapsto \phi_V\left(\phi_U^{-1}(q)\right). \end{aligned}$$

i) Show that the tangent vector to the curve, with respect to the local trivialization (V, ϕ_V) is

$$\dot{\tilde{\gamma}}_V = \mathrm{d}\psi_{U,V}\big|_{(\phi_U(p))}(\dot{\tilde{\gamma}}_U).$$

(*Hint*: use the chain rule.)

ii) Writing the tangent vector $\dot{\tilde{\gamma}}_U$ as $\begin{pmatrix} \dot{t}_U \\ \dot{x}_U \end{pmatrix}$ with $\dot{t}_U \in \mathbb{R}$, $\dot{x}_U \in \mathbb{R}^3$, show that

$$\mathrm{d}\tilde{\pi}_U\big|_{\phi_U(p)}\left(\dot{\tilde{\gamma}}_U\right) = \dot{t}_U.$$

Using the fact that the projection map commutes with local trivializations, show also that

$$\dot{t}_U = \dot{t}_V,$$

so that the first component of the tangent vector is *independent of the local trivialisation*. Hence show that

$$\mathrm{d}\psi_{U,V}\big|_{\phi_U(p)} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R_{U,V}(p) \end{array}\right),$$

where, for each p, $R_{U,V}(p)$ is a linear map from $\mathbb{R}^3$ to itself.

d) A connection labels certain curves through $\mathbb{G}$ as "straight lines". Suppose that, in some local trivialization (U, ϕ_U), these straight lines really are straight lines: that is, any curve with a constant tangent vector is declared a "straight line". In other words, if

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\dot{\tilde{\gamma}}_U(\tau) = \ddot{\tilde{\gamma}}_U(\tau) = 0,$$

then γ is a "straight line". Suppose also that with respect to this trivialization the inner product between two tangent vectors $\dot{\gamma}$ and $\dot{\xi}$

$$\langle \dot{\gamma}, \dot{\xi} \rangle = \dot{\vec{\gamma}}_U \cdot \dot{\vec{\xi}}_U,$$

where the dot represents the usual dot product in $\mathbb{R}^3$.

i) Show that, in another local trivialization (V, ϕ_V), the inner product is also given by the dot product if

$$\mathrm{d}\psi_{U,V}\big|_{\phi_U(p)} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R(p) \end{array}\right),$$

where for each p, $R(p) \in O(3)$ (if orientation is also preserved then $R \in SO(3)$). Show that straight lines are also given in the trivialization (V, ϕ_V) by curves with constant tangent vectors if and only if the matrix R is constant.

ii) Hence show that, if the local trivialization (V, ϕ_V) preserves both the inner product and the "straight lines", then $\psi_{U,V}$ is a Galilean transformation: choosing some arbitrary origin for $\mathbb{E} \times \mathbb{E}^3$ so that points can be represented by points in $\mathbb{R} \times \mathbb{R}^3$, we have

$$\psi_{U,V} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t + t_0 \\ Rx + x_0 \end{pmatrix}.$$

All of the local trivializations which are related to (U, ϕ_U) by such a transformation are therefore called *inertial frames*.

iii) Now suppose that there is another local trivialization (W, ϕ_W) which preserves the inner product but does not necessarily preserve "straight lines". Show that, in this trivialization, "straight lines" instead satisfy

$$\ddot{t}_W = 0$$

$$\ddot{x}_W = \dot{R}R^T \dot{x}_W.$$

By writing $R(\gamma(\tau))$ as $\left(R \circ \phi_W^{-1} \circ \phi_W \circ \gamma\right)$, show that we can write

$$\dot{R}(\tau) = \left(\Gamma_t(\dot{t}_W) + \Gamma_x(\dot{x}_W)\right) R,$$

where γ_t and γ_x are linear maps, with

$$\begin{aligned} \Gamma_t &: \mathbb{R} \to M \\ \Gamma_x &: \mathbb{R}^3 \to M \\ M &= \{\text{skew symmetric } 3 \times 3 \text{ matrices}\}. \end{aligned}$$

Thus, with respect to a general trivialization which preserves the inner product, straight lines are given by

$$\begin{aligned} \ddot{t}_W &= 0 \\ \ddot{x}_W &= \Gamma_t(\dot{t}_W)\dot{x}_W + \Gamma_x(\dot{x}_W)\dot{x}_W. \end{aligned}$$

Γ is called the connection. More precisely, the connection is the object which vanishes in the original trivialization (U, ϕ_U), and which is given by Γ_t and Γ_x in the trivialization (W, ϕ_W).