Problem sheet 3

Joe Keir

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All questions use natural units, where Newton's constant G and the speed of light c are both equal to 1.

Questions marked with a star * are optional extension questions which go beyond the scope of the course. They will not be discussed in class unless all other questions have already been covered. You are advised to only attempt these questions if you have already completed the other questions on the sheet.

Questions/comments/correction to Joseph.Keir@maths.ox.ac.uk

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1. *Symmetries of the Riemann tensor*

The Riemann tensor (associated to a metric g) satisfies the following algebraic identities:

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= R_{\rho\sigma\mu\nu} \\ R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma} \\ R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} &= 0 \end{aligned}$$

a) Show that, in four spacetime dimensions, the Riemann tensor has 20 independent components. *(You might prefer to attempt part b) first, and then set $d = 4$)*

b) Show that, in d spacetime dimensions, the Riemann tensor has $\frac{1}{12}d^2(d^2-1)$ independent components.

(Hint: working in components in some coordinate system, the third symmetry in the list above gives a new relation between the components of the Riemann tensor if and only if each the indices takes a distinct value – if two of the indices take the same value, then this third symmetry is implied by the other two.

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2. *The Riemann tensor in 2 and 3 dimensions*

a) i) Show that, in two dimensions, the Riemann tensor is given by

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}R\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$

(Hint: use question 1!)

ii) Hence show that *all* two-dimensional spacetimes satisfy the vacuum Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$$

b) i) Show that, in three spacetime dimensions, the Riemann tensor and the Ricci tensor have the same number of independent components. Hence show that the Riemann tensor is given by

$$R_{\mu\nu\rho\sigma} = R_{\nu\sigma}g_{\mu\rho} + R_{\mu\rho}g_{\nu\sigma} - R_{\nu\rho}g_{\mu\sigma} - R_{\mu\sigma}g_{\nu\rho} - \frac{1}{2}R\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$

ii) Now suppose that the metric g solves the Einstein equations in three spacetime dimensions. Use part **b) i)** to write down an expression for the Riemann tensor in terms of the energy momentum tensor and the metric.

If the Riemann tensor vanishes in some region, then there is a theorem that the spacetime is *locally isometric to Minkowski space*. That is, there is some subset of the region where the Riemann tensor vanishes, and a map from that region to Minkowski space preserving the metric.

Suppose that we have a solution to the three dimensional Einstein equations. Show that, if the matter vanishes in some region, then that region is locally isometric to Minkowski space.

(It is sometimes said that "there are no gravitational dynamics in 3D"*. From the results above we see that, in 3D, the metric does not do anything interesting on its own – in the absence of matter, it just looks like Minkowski space, and in the presence of matter the metric just 'reacts' to the energy momentum density. The picture is very different in four spacetime dimensions, where there are gravitational waves (even in the absense of matter), and in the presence of matter there are both gravitational and matter degrees of freedom.)*

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3. *Wave equations*

a) A scalar field ϕ satisfies the equation

$$\nabla^\mu \nabla_\mu \phi = 0$$

The operator $\nabla^\mu\nabla_\mu$ is sometimes written $\Box_g$, and is called the *(geometric) wave operator.*

Show that, in normal coordinates at a point p, the field ϕ satisfies

$$\left(\left(-\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2\right)\phi \right)\Big|_p = 0$$

(This explains how the operator $\Box_g$ *relates to the standard wave operator).*

b) Suppose that the Maxwell field $F_{\mu\nu}$ can be written as

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

In this case, the covector field A is called the *four-potential.*

i) Show that the first part of Maxwell's equations is automatically satisfied:

$$\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0$$

ii) Show that the equations are invariant under the *gauge transformation*

$$A_\mu \mapsto A_\mu + (\mathrm{d}f)_\mu$$

for any smooth function f.

iii) Gauge transformations can be used to ensure that the four-potential satisfies the *Lorentz gauge condition*

$$\nabla^\mu A_\mu = 0$$

In this case, use the other Maxwell equation

$$\nabla^\mu F_{\mu\nu} = J_\nu$$

to show that the four-potential satisfies the wave equation

$$\Box_g A_\mu - R_\mu{}^\nu A_\nu = J_\mu \tag{1}$$

iv) Recall that the curvature of a manifold depends on the first two derivatives of the metric. Thus the 'wave equation' (1) has the undesirable property that the highest order terms (often called the *principal terms* in PDE) do not consist simply of the wave operator acting on the four-potential.

Explain how, if the Einstein equations are satisfied, equation (1) can be transformed into one in which the principal terms *are* just the wave operator acting on the four-potential. Write down the resulting nonlinear wave equation for the four-potential. You will need the energy-momentum tensor for a Maxwell field, which is given by

$$T_{\mu\nu} = F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

c) i) Show that, in the vacuum ($T_{\mu\nu} = 0$), the Riemann tensor is 'divergence free':

$$\nabla^\mu R_{\mu\nu\rho\sigma} = 0$$

ii) Hence show that, in the vacuum, the Riemann tensor satisfies the nonlinear wave equation

$$\Box_g R_{\mu\nu\rho\sigma} + 2R_\mu{}^\alpha{}_\nu{}^\beta R_{\alpha\beta\rho\sigma} - 2R_\mu{}^\alpha{}_\sigma{}^\beta R_{\nu\alpha\beta\rho} + 2R_\mu{}^\alpha{}_\rho{}^\beta R_{\nu\alpha\sigma\beta} = 0$$

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4. *Jacobi fields on constant curvature manifolds*

A *spacetime with constant curvature* is one where the Riemann tensor takes the form

$$R_{\mu\nu\rho\sigma} = K\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$

(c.f. the Riemann tensor in two dimensions), where K is a *constant* scalar function.

a) Suppose that the spacetime is four dimensional. Write the scalar K in terms of the Ricci scalar R.

b) i) Now suppose that the Ricci scalar is negative. Write down an expression for the geodesic deviation equation in this case, and give its general solution. Give a physical interpretation of this result.

ii) Repeat step **b) i)** in the cases of *positive* Ricci scalar, and zero Ricci scalar.

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5. *The Shapiro time delay*

A classic test of general relativity (along with gravitational redshift, the perihelion precession of Mercury and the gravitational bending of light) is the time delay of light signals bounced off satellites or planets, when those light signals travel close to the sun.

For this test, we work in the approximation where the path of the light ray is given (in Schwarzschild coordinates) by

$$r \sin\phi = b$$

where b is a constant called the *impact parameter.* The light ray will also be taken to lie entirely in the equatorial plane ($\theta = \frac{\pi}{2}$).

a) Show that, in this case, along the light ray the coordinates r and ϕ satisfy

$$r^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 = \frac{b^2}{r^2 - b^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2$$

where λ is *any* parameter along the light ray.

b) Hence show that along this light ray, neglecting terms of order $\mathcal{O}\left(\left(\frac{M}{r}\right)^2\right)$, we have

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\frac{r}{\sqrt{r^2-b^2}}\left(1+\frac{2M}{r}-\frac{Mb^2}{r^3}\right)+\mathcal{O}\left(\left(\frac{M}{r}\right)^2\right)$$

c) Finally, show that, in this approximation, the (coordinate) time taken for a light ray to travel from $r=b$ to $r=r_1$ (where $r_1 > b$) is

$$\Delta t = \sqrt{r_1^2-b^2} + 2M\mathrm{arcosh}\left(\frac{r_1}{b}\right) - \frac{M}{r_1}\sqrt{r_1^2-b^2}$$

and give a physical interpretation of the first term. *The remaining terms give the* Shapiro time delay*, a delay in the time taken for light rays to travel through a gravitational field predicted by general relativity. This delay has been measured experimentally.*

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6. *Proper acceleration*

Let $\gamma(\tau)$ be a timelike curve parametrised by proper time, and let $\dot{\gamma}(\tau)$ be the tangent vector to the curve at the point $\gamma(\tau)$. Then the *proper acceleration* of the curve γ at the point $\gamma(\tau)$ is defined as the vector

$$(\ddot{\gamma}(\tau))^\mu = \left(\dot{\gamma}^\nu \nabla_\nu \dot{\gamma}^\mu\right)\big|_\tau$$

a) Show that the proper acceleration of a timelike curve is always orthogonal to the tangent to the curve, i.e.

$$g\left(\ddot{\gamma}(\tau), \dot{\gamma}(\tau)\right) = 0$$

b) i) Calculate the proper acceleration of an observer who remains at a *constant* value of r, say $r=r_0$, in the Schwarzschild spacetime. Note in particular that, despite the fact that the spatial coordinates of this observer remain constant, they do accelerate!

ii) Calculate the magnitude of the acceleration of this observer, that is, $\sqrt{g(\ddot{\gamma}(\tau), \ddot{\gamma}(\tau)}$. This is the magnitude of the force necessary to keep the observer stationary.

iii) Conspiracy theorists called "Flat Earthers" sometimes claim that we are not at rest on an Earth with a force of gravity acting on us, but instead the (flat) surface of the Earth is accelerating upwards, and this is what we experience as the force of gravity. Comment on the accuracy of this remark.

iv) What happens as $r_0 \to 2M$? Could a physical observer, equipped with a very powerful rocket, stay at $r_0 = 2M$?

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7. *Radial geodesics and escape velocity*

Consider a massive particle moving along a radial geodesic in the Schwarzschild spacetime, emitted in an outgoing direction from the surface $r=R$.

a) Define

$$\frac{E}{m} := \left(1-\frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}$$

where τ is the proper time along the geodesic. Show explicitly that $\frac{E}{m}$ is constant along the geodesic. Physically, this constant represents the *energy per unit mass* of the particle.

b) Let $\frac{p}{m}$ be the coordinate radial 'velocity', i.e.

$$p := m\frac{\mathrm{d}r}{\mathrm{d}\tau}$$

Show that Einstein's famous formula $E^2 = p^2 + m^2$ (which, with the speed of light c restored and in the rest frame of the particle, so $p = 0$, reduces to $E = mc^2$) gets an additional correction in the Schwarzschild metric:

$$E^2 = p^2 + m^2 \left(1 - \frac{2M}{r}\right)$$

c) Let v be the initial coordinate velocity of the particle, i.e.

$$v := \left.\frac{\mathrm{d}r}{\mathrm{d}\tau}\right|_{r=R}$$

Show that the maximum value of r that is obtained along the path of the particle is

$$r_{\max} = \frac{2M}{\frac{2M}{R} - v^2}$$

Compare this to the Newtonian value (see example sheet 0). What happens if $v^2 > \frac{2M}{R}$?

d) The coordinate r and the associated coordinate velocity $\frac{\mathrm{d}r}{\mathrm{d}\tau}$ do not have any physical meaning in GR. To get a handle on what a local observer would actually see, we can instead work in normal coordinates.

Let the event where the particle is ejected from $r = R$ be the spacetime point p. Suppose a local observer at p sets up normal coordinates $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ centred at p, so that p has coordinates $(0, 0, 0, 0)$.

i) Show that it is possible to choose these coordinates so that

$$\left.\mathrm{d}\tilde{t}\right|_p := \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} \mathrm{d}t$$

$$\left.\mathrm{d}\tilde{x}\right|_p := \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} \mathrm{d}r$$

and so that the components of the velocity of the particle in the $\tilde{y}$ and $\tilde{z}$ directions vanish (*Hint: you do not need to actually construct these coordinates, it is enough to show that their differentials at the point p obey the required conditions.*).

ii) Hence show that the initial velocity of the particle, in these coordinates, is

$$\tilde{v} = \left.\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{t}}\right|_{\tilde{t}=0} = \frac{vm}{E} = \frac{v}{\sqrt{v^2 + \left(1 - \frac{2M}{r}\right)}}$$

iii) Now write $r_{\max}$ in terms of $\tilde{v}$ instead of v. It is sometimes said that "gravity in GR is stronger than Newtonian gravity". Comment on this statement.

***8.** *The Schwarzschild metric solves Einstein's equations*

Show explicitly (using Schwarzschild coordinates) that the Schwarzschild metric solves the vacuum Einstein equations $R_{\mu\nu} = 0$. This is a very long calculation and should be done in your own time – there is no need to hand in a solution. You may find it helpful to go through the following steps:

a) Write out all the Christoffel symbols. You may find it quicker to use the Euler-Lagrange equations for geodesics than to use the expression for Christoffel symbols in terms of derivatives of the metric.

b) Use the following expression to calculate the components of the Riemann curvature tensor:

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{bc}$$

c) Contract the indices a and c in the expression above to find the components of the Ricci curvature tensor R_{bd}, and check that these all vanish.