Problem sheet 4

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All questions use natural units, where Newton's constant G and the speed of light c are both equal to 1.

Questions marked with a star * are optional extension questions which go beyond the scope of the course. They will not be discussed in class unless all other questions have already been covered. You are advised to only attempt these questions if you have already completed the other questions on the sheet.

Questions/comments/correction to Joseph.Keir@maths.ox.ac.uk

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1. *Eddington-Finkelstein coordinates and the event horizon.*

a) Write down the Schwarzschild metric in ingoing Eddington-Finkelstein coordinates. Also write down the inverse metric, and explain why these coordinates allow us to deal with the region $r < 2M$.

b) Consider a null geodesic travelling in the Schwarzschild spacetime, parametrised by an affine parameter s. Working in ingoing Eddington-Finkelstein coordinates, show that:

i) If $\theta = \frac{\pi}{2}$ initially, and $\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0$ initially, then $\theta = \frac{\pi}{2}$ forever.

ii) There is a conserved quantity due to the fact that the Lagrangian is independent of the coordinate v. Write down an expression for this quantity.

iii) There is a conserved quantity due to the fact that the Lagrangian is independent of the coordinate ϕ. Write down an expression for this quantity.

iv) Using the fact that the tangent to this geodesic is null, write down an expression for $\frac{\mathrm{d}r}{\mathrm{d}s}$ in terms of the coordinate r and the conserved quantities derived above, assuming that the geodesic lies entirely in the equatorial plane $\theta = \frac{\pi}{2}$.

c) Show, using Eddington-Finkelstein coordinates, that the curve $(v, r, \theta, \phi) = (v(s), 2M, \theta_0, \phi_0)$ (for constants θ_0 and ϕ_0) is a null geodesic. How does the coordinate v along this geodesic relate to the affine parameter s? *(Hint: look at the Euler-Lagrange equation associated with varying r).*

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2. *Inevitable doom inside a black hole*

Bob is falling into a black hole and, unfortunately for them, they have just passed the event horizon at $r = 2M$.

a) Write down (in Schwarzschild coordinates) the metric in the interior of the Schwarzschild black hole. Which of the coordinates corresponds to time?

b) Show that, along *any* timelike curve (not necessarily a geodesic), the function r decreases as proper time increases, with the following bound on the rate:

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} \leq -\sqrt{\frac{2M}{r} - 1}$$

c) Hence show that the maximum proper time along the curve between crossing the event horizon and reaching the singularity at $r = 0$ is

$$\tau = M\pi$$

Hint: you may find the substitution $u = \sqrt{1 - \frac{r}{2M}}$ *helpful.*

From Bob's point of view, is it better to fall into a large black hole or a small one?

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3. *Isotropic coordinates.*

Beginning in the usual Schwarzschild coordinates (t, r, θ, ϕ), define a function ρ by

$$r := \rho \left(1 + \frac{M}{2\rho}\right)^2$$

Now define functions x, y and z as

$$\begin{aligned} x &:= \rho \sin\theta \cos\phi \\ y &:= \rho \sin\theta \sin \sin \\ z &:= \rho \cos\theta \end{aligned}$$

a) Write down the Schwarzschild metric in *isotropic coordinates* (t, x, y, z) *(Hint: you might find it helpful to first write down the metric in the coordinates* (t, ρ, θ, ϕ)*).* What is the range of the coordinate ρ? What happens to the metric at the (inner) boundary of this coordinate range?

b) The *light cone* is a structure in the tangent space at every point in a Lorentzian manifold $(\mathcal{M}, g)$, defined as follows:

$$T_p(\mathcal{M}) \supset C_p(\mathcal{M}) = \big\{X \in T_p(\mathcal{M}) \,\big|\, g(X, X) = 0\big\}$$

Show that, working in isotropic coordinates, the light cones are isotropic (hence the name). In other words, show that there is a scalar field $f : \mathcal{M} \to \mathbb{R}$ such that

$$C_p(\mathcal{M}) = \big\{X \in T_p(\mathcal{M}) \;\big|\; (X^0)^2 = f(p)\left((X^1)^2 + (X^2)^2 + (X^3)^2\right)\big\}$$

so that, from the point of view of these coordinates, the light cones look the same in every spatial direction at every point.

c) Now concentrate on the spatial part of the metric, i.e. drop the part of the metric that is proportional to $\mathrm{d}t^2$. This defines a Riemannian metric on the surfaces of constant t (sometimes it is said that the Lorentzian metric g *induces* a Riemannian metric on the hypersurfaces of constant t). Let's call this spatial part of the metric h.

The *angle* θ between two vectors X and Y on a Riemannian manifold (both lying in the same tangent space, i.e. at a single point on the Riemannian manifold) is defined by the formula

$$\cos\theta = \frac{h(X, Y)}{\sqrt{h(X, X) h(Y, Y)}}$$

Show that there is a map from a surface of constant t in Schwarzschild to $\mathbb{R}^3$ which preserves the angles between all vectors (the angles between vectors in $\mathbb{R}^3$ being defined in the standard way). A map of this type is called a *conformal map* or *conformal isometry.*

4. *Stability of the Einstein static universe*

Write down the Friedman equations with $k = 1$ (a closed universe), $\Lambda \neq 0$ and where the matter content is given by dust ($p = 0$).

a) Show that there is a single value of the scale factor $a = a_0$ which leads to a static universe ($\dot{a} = \ddot{a} = 0$). Give this value of a in terms of the cosmological constant. Give the matter density $\rho = \rho_0$ needed to support this universe (also in terms of Λ).

b) Now consider a small perturbation around this solution: set

$$a = a_0 + \epsilon a_1(\tau)$$
$$\rho = \rho_0 + \epsilon \rho_1(\tau)$$

and expand the Friedmann equations to leading order in ϵ. Show that the general solutions to these equations have the magnitudes of a_1 and ρ_1 growing exponentially quickly, and so the Einstein static universe is unstable.

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5. *Conformal time and worldlines in an FLRW spacetime*

a) Define a coordinate η by

$$\mathrm{d}\tau = a\mathrm{d}\eta$$

Write down the general form of the Robertson-Walker metric in coordinates (η, r, θ, ϕ). *(The coordinate η is called the* conformal time*).*

b) Now consider the closed case ($k = 1$). Define yet another coordinate, χ by

$$\sin \chi = r$$

and write down the Robertson-Walker metric in the coordinates $(\eta, \chi, \theta, \phi)$.

i) Comment on the spatial part of the metric.

ii) Show that there are null geodesics lying entirely within the "equatorial hyperplane" $\chi = \theta = \frac{\pi}{2}$. Show that, along such a geodesic,

$$\frac{\mathrm{d}\phi}{\mathrm{d}\eta} = \pm 1$$

c) Suppose that this describes a *radiation dominated universe*. Write down the equation of state in this case, and show that the matter density is related to the scale factor in this case by

$$\rho = \rho_0 a^{-4}$$

where ρ_0 is a constant.

d) Use the Friedmann equations to show that, if the cosmological constant Λ vanishes (and in the radiation dominated case with $k = 1$) the scale factor obeys the equation

$$3\left(\frac{\mathrm{d}a}{\mathrm{d}\eta}\right)^2 + 3a^2 = 8\pi\rho_0$$

and solve this equation with the initial data $a = 0$ when $\eta = 0$. What value does the conformal time take when the "big crunch" occurs?

e) What proportion of the circumference of the universe can a photon (moving on the equatorial hyperplane) traverse before the end of time?

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6. *De Sitter space and cosmological horizons*

De Sitter space can be defined as a submanifold of *5-dimensional* Minkowski space $\mathbb{M}^5$, with the Minkowski metric

$$g_5 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 + (\mathrm{d}x^4)^2$$

Consider the hyperboloid $-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = \frac{3}{\Lambda}$, where $\Lambda > 0$. De Sitter space consists of this hyperboloid, equipped with the metric found by restricting the five dimensional ambient metric g_5 to the hyperboloid (in other words, the metric on De Sitter space is the metric induced by the ambient metric g_5).

a) Define the coordinates $(\tau, \rho, r, \theta, \phi)$ on $\mathbb{M}^5$ as follows:

$$\begin{aligned}
x^0 &= \rho\left(\sqrt{\frac{3}{\Lambda}}\sinh\left(\sqrt{\frac{\Lambda}{3}}\tau\right) + \frac{1}{2}\sqrt{\frac{\Lambda}{3}}r^2 e^{\sqrt{\frac{\Lambda}{3}}\tau}\right)\\
x^1 &= \rho\left(\sqrt{\frac{3}{\Lambda}}\cosh\left(\sqrt{\frac{\Lambda}{3}}\tau\right) - \frac{1}{2}\sqrt{\frac{\Lambda}{3}}r^2 e^{\sqrt{\frac{\Lambda}{3}}\tau}\right)\\
x^2 &= \rho e^{\sqrt{\frac{\Lambda}{3}}\tau} r \sin\theta\cos\phi\\
x^3 &= \rho e^{\sqrt{\frac{\Lambda}{3}}\tau} r \sin\theta\sin\phi\\
x^4 &= \rho e^{\sqrt{\frac{\Lambda}{3}}\tau} r \cos\theta
\end{aligned}$$

i) Write out the 5-dimensional Minkowski metric in these coordinates.

ii) Show that the hyperboloid $-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = \frac{3}{\Lambda}$ is given by $\rho = 1$ in the coordinates above. Hence write down the induced metric on the hyperboloid (i.e. set $\rho = 1$ and drop all terms involving $\mathrm{d}\rho$).

iii) Show that this metric is of the standard Robertson-Walker form, and identify the scale factor $a(\tau)$. Is this metric of *flat*, *open* or *closed* form?

b) Many cosmological models posses a *particle horizon* or *cosmological horizon*. Unlike black hole *event horizons*, a cosmological horizon is personal – different observers will label different surfaces as their own individual cosmological horizon.

Let $\gamma(\tau)$ be a worldline of an observer, i.e. a timelike curve through De Sitter space. The particle horizon at time τ is the boundary of the region from which signals can reach the observer at τ, so, locally, it looks like the past light cone of the point $\gamma(\tau)$ (see figure 1).

i) Define a conformal time coordinate η by

$$\mathrm{d}\tau = e^{\sqrt{\frac{\Lambda}{3}}\tau}\mathrm{d}\eta$$

and write down the metric in (η, r, θ, ϕ) coordinates, choosing η so that $\eta = 0$ when $\tau = 0$.

ii) Consider an observer moving along the worldline $r = 0$, i.e. an observer who stays at the (spatial) origin. By considering the form of the metric in the coordinates (η, r, θ, ϕ), show that the cosmological horizon at the conformal time $\eta = \eta_0$ is given by the surface $r = \eta_0 - \eta$. You may assume that the boundary of the set of events from which causal signals can reach this observer is given by the set of past-directed radial null geodesics from the point $\eta = \eta_0$, $r = 0$.

iii) Write the position of the cosmological horizon as a function of τ and τ_0, where $\tau_0 = \tau(\eta_0)$. What happens as $\tau_0 \to \infty$? Show that there are some spacetime events that the observer will *never* see – these events are said to lie outside this observer's *final horizon.*

This is actually the final fate of the universe, according to the best current cosmological models. So, even if you were immortal, you still wouldn't be able to see everything in the universe!

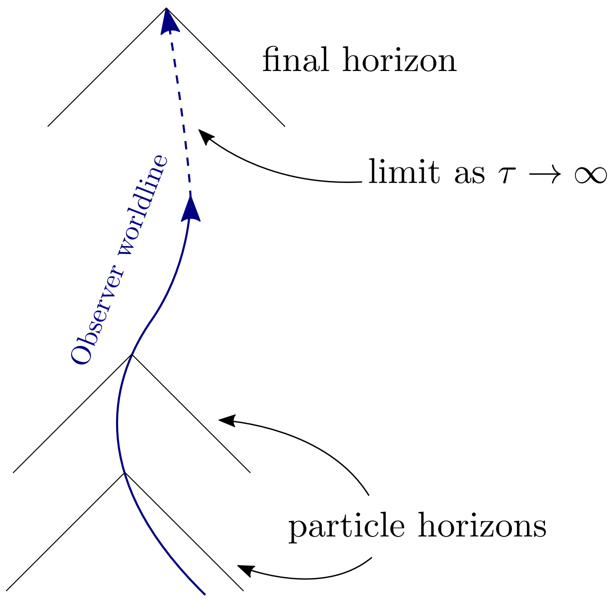

Figure 1: An observer moves along the blue worldline in a cosmological spacetime. Two particle horizons are shown: the coordinates are such that the light rays travel at a 45 degree angle.
Now imagine that we choose coordinates such that the entire, infinite future of the blue worldline is compressed so that it fits into this finite diagram. This means that the dotted part of the worldline is traversed in an infinite proper time. We can consider the limit of the particle horizons as the proper time $\tau \to \infty$ – this is called the *final horizon*.

***7.** *The Schwarzschild metric using frame fields*

This question will guide you through an alternative approach to showing that the Schwarzschild metric solves the Einstein equations. This approach is based on the idea of a *frame field*, which provides an alternative to coordinates as a basic object for manipulating expressions in general relativity.

Using Schwarzschild coordinates (t, r, θ, ϕ), define the one-forms (or covectors)

$$\begin{aligned} f^0 &:= \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \mathrm{d}t \\ f^1 &:= \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} \mathrm{d}r \\ f^2 &:= r\mathrm{d}\theta \\ f^3 &:= r\sin\theta \mathrm{d}\phi \end{aligned}$$

We'll use capitol Latin letters $A, B, C \dots$ to label these one-forms, i.e. we write them as ω^A.

a) Find the dual basis of vector fields e_A, where $f^A(e_B) = \delta^A_B$. Show also that

$$(e_A)^\sharp = m_{AB} f^B$$

b) Show that the metric is given by

$$g = m_{AB} f^A f^B$$

where $m_{AB} = \mathrm{diag}(-1, 1, 1, 1)$. Likewise, show that the inverse metric is given by

$$g^{-1} = (m^{-1})^{AB} e_A e_B$$

Before outlining the next part of this approach, we need to be familiar with *two-forms*. These objects were outlined in question 8 of problem sheet 2, but, to summarize,

- A two-form is an *antisymmetric* rank $(0,2)$ tensor field.
- Given two one-forms (i.e. covectors) η, μ, we can form a two-form $\eta \wedge \mu$, using the "wedge product". This two-form acts on a pair of vectors X and Y as follows:

$$\eta \wedge \mu(X,Y) = \eta(X)\mu(Y) - \mu(X)\eta(Y)$$

  or if you prefer components

$$(\eta \wedge \mu)_{ab} = \eta_a \mu_b - \mu_a \eta_b$$

- Given coordinates x^a, a basis of two forms is given by the tensor fields $\mathrm{d}x^a \wedge \mathrm{d}x^b$ with $a > b$.
- Given a one-form (or covector field) η, we can define the two-form $\mathrm{d}\eta$. In terms of the basis given above, this can be written as follows:

$$\eta = \eta_a \mathrm{d}x^a$$
$$\mathrm{d}\eta = \frac{\partial \eta_a}{\partial x^b} \mathrm{d}x^b \wedge \mathrm{d}x^a$$

  Alternatively, in coordinate-independent notation, we have

$$\mathrm{d}\eta(X,Y) = X\left(\eta(Y)\right) - Y\left(\eta(X)\right) - \eta\left([X,Y]\right)$$

  where X and Y are vector fields, and $[X,Y]$ is their commutator.

The next step in our process is to find the *connection coefficients* ω_{ab} (also known as Ricci rotation coefficients or connection one-forms). These can be viewed as the components of an *antisymmetric matrix of one-forms*, or equivalently as a set of 6 one-forms (or covectors) labelled by a pair of antisymmetric indices.

The connection coefficients are defined by the relationship

$$m_{AB} \mathrm{d}f^B = -\omega_{AB} \wedge f^B \tag{1}$$

We first want to show that this equation has a unique solution.

c) Suppose that there is another set of one forms (labelled by antisymmetric indices) $\bar{\omega}_{AB}$ satisfying

$$m_{AB} \mathrm{d}f^B = -\bar{\omega}_{AB} \wedge f^B$$

Writing $\omega_{AB} - \bar{\omega}_{AB} = \eta_{ABC} f^C$, show that

i) $\eta_{ABC} = -\eta_{BAC}$

ii) $\eta_{ABC} = \eta_{ACB}$

iii) Hence conclude that $\eta_{ABC} = 0$, so $\omega = \bar{\omega}$ and the solution to equation (1) is unique.

Next we want to actually find the connection coefficients. We claim that they are given by the formula

$$\omega_{AB} = -g(\nabla_C e_A, e_B) f^C = -g(e^\mu_C \nabla_\mu e_A, e_B) f^C$$

To show this, we can act with $m_{AB}\mathrm{d}f^B$ and with $-\omega_{AB} \wedge f^B$ on a basis. Before we do that, we can do the following exercise:

d) Use the fact that $g(e_A, e_B) = m_{AB}$ to show that

$$g(\nabla_C e_A, e_B) = -g(\nabla_C e_B, e_A)$$

Now, returning to the claim that $\omega_{AB} = -g(\nabla_C e_A, e_B) f^C$, we can compute

$$\begin{aligned}
m_{AB} \mathrm{d} f^B(e_C, e_D) &= m_{AB} \left(e_C(f^B(e_D)) - e_D(f^B(e_C)) - f^B([e_C, e_D]) \right) \\
&= m_{AB} \left(e_C(\delta^D_B) - e_D(\delta^B_C) - f^B([e_C, e_D]) \right) \\
&= -m_{AB} f^B([e_C, e_D]) \\
&= m_{AB} f^B \left(\nabla_D e_C - \nabla_C e_D \right) \\
&= g\left(e_A, \nabla_D e_C - \nabla_C e_D \right)
\end{aligned}$$

On the other hand, using the claimed expression for ω_{AB}, we can compute

$$\begin{aligned}
-\omega_{AB} \wedge f^B(e_C, e_D) &= g(\nabla_E e_A, e_B) f^E \wedge f^B(e_C, e_D) \\
&= g(\nabla_E e_A, e_B)(\delta^E_C \delta^B_D - \delta^E_D \delta^B_C) \\
&= -g(\nabla_D e_A, e_C) + g(\nabla_C e_A, e_D) \\
&= g(\nabla_D e_C, e_A) - g(\nabla_C e_D, e_A)
\end{aligned}$$

where in the last line we have used the antisymmetry established in part **d)**.

e) Compute the connection coefficients associated to the orthonormal frame e^A in Schwarzschild. There are several ways to do this:

e) Approach 1

We can try the direct approach, using the formula $\omega_{AB} = -g(\nabla_C e_A, e_B) f^C$. We can compute the quantity $g(\nabla_C e_A, e_B)$ without computing any Christoffel symbols using the following trick:

$$\begin{aligned}
g(\nabla_C e_A, e_B) &= g([e_C, e_A], e_B) + g(\nabla_A e_C, e_B) \\
&= g([e_C, e_A], e_B) - g(\nabla_A e_B, e_C) \\
&= g([e_C, e_A], e_B) - g([e_A, e_B], e_C) - g(\nabla_B e_A, e_C) \\
&= g([e_C, e_A], e_B) - g([e_A, e_B], e_C) + g(\nabla_B e_C, e_A) \\
&= g([e_C, e_A], e_B) - g([e_A, e_B], e_C) + g([e_B, e_C], e_A) + g(\nabla_C e_B, e_A) \\
&= g([e_C, e_A], e_B) - g([e_A, e_B], e_C) + g([e_B, e_C], e_A) - g(\nabla_C e_A, e_B) \\
\Rightarrow g(\nabla_C e_A, e_B) &= \frac{1}{2} \left(g([e_C, e_A], e_B) - g([e_A, e_B], e_C) + g([e_B, e_C], e_A) \right)
\end{aligned}$$

This allows us to compute $g(\nabla_C e_A, e_B)$ without computing Christoffel symbols – instead, we just need to compute the commutators $[e_A, e_B]$.

e) Approach 2

The alternative to the direct approach is to simply write out the formula

$$m_{AB} \mathrm{d} f^B = -\omega_{AB} \wedge f^B$$

The left hand side is easy to compute. Then we just look for a matrix of one-forms ω_{AB} satisfying the equation above. If we can find such a matrix, then we know that it is unique, so try to guess the answer! In practice this is often a much quicker approach.

Next we want to relate the connection coefficients to the curvature. We claim that the components of the Riemann tensor are

$$R_{ABCD} f^C \wedge f^D = \mathrm{d}\omega_{AB} + \omega_{AC} \wedge \omega^C{}_B$$

To prove this claim, we can act with $\mathrm{d}\omega_{AB}$ on a basis:

$$
\begin{aligned}
\mathrm{d}\omega_{AB}(e_C, e_D) &= e_C(\omega_{AB}(e_D)) - e_D(\omega_{AB}(e_C)) - \omega_{AB}([e_C, e_D]) \\
&= -e_C\left(g(\nabla_D e_A, e_B)\right) + e_D\left(g(\nabla_C e_A, e_B)\right) + g(\nabla_{[e_C, e_D]} e_A, e_B) \\
&= -g(\nabla_C \nabla_D e_A, e_B) + g(\nabla_D \nabla_C e_A, e_B) + g(\nabla_{[e_C, e_D]} e_A, e_B) - g(\nabla_D e_A, \nabla_C e_B) \\
&\quad + g(\nabla_C e_A, \nabla_D e_B) \\
&= -R_{BACD} - g(\nabla_D e_A, \nabla_C e_B) + g(\nabla_C e_A, \nabla_D e_B) \\
&= R_{ABCD} - g(\nabla_D e_A, e_E)(m^{-1})^{EF} g(\nabla_C e_B, e_F) + g(\nabla_C e_A, e_E)(m^{-1})^{EF} g(\nabla_D e_B, e_F) \\
&= R_{ABCD} - (m^{-1})^{EF}(\omega_{AE})_D(\omega_{BF})_C + (m^{-1})^{EF}(\omega_{AE})_C(\omega_{BF})_D \\
&= R_{ABCD} + (m^{-1})^{EF}\left(\omega_{AE} \wedge \omega_{BF}\right)_{CD}
\end{aligned}
$$

So in other words, if we define the *curvature two-form* Ω_{AB} (labelled by a pair of antisymmetric matrices) as

$$\Omega_{AB} = \mathrm{d}\omega_{AB} + (m^{-1})^{CD}\omega_{AC} \wedge \omega_B{}^D = \mathrm{d}\omega_{AB} + \omega_{AC} \wedge \omega^C{}_B$$

then we have

$$R_{ABCD} = (\Omega_{AB})_{CD} = \Omega_{AB}(e_C, e_D)$$

f) Now, compute the curvature two-form Ω_{AB} for the Schwarzschild metric. Note how much easier it is to compute $\mathrm{d}\omega_{AB}$ and $\omega_{AC} \wedge \omega^C{}_B$ compared with computing the curvature using coordinates and Christoffel symbols!

g) Now that the components of the Riemann tensor R_{ABCD} have been computed in an *orthonormal frame*, we can easily compute the components of the Ricci tensor: it is

$$R_{AB} = (m^{-1})^{CD} R_{ACBD} = -R_{A0B0} + R_{A1B1} + R_{A2B2} + R_{A3B3}$$

Use this calculation to show that the Schwarzschild metric satisfies the Einstein equations.