C7.5 Lecture 2: Spacetime

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Overview

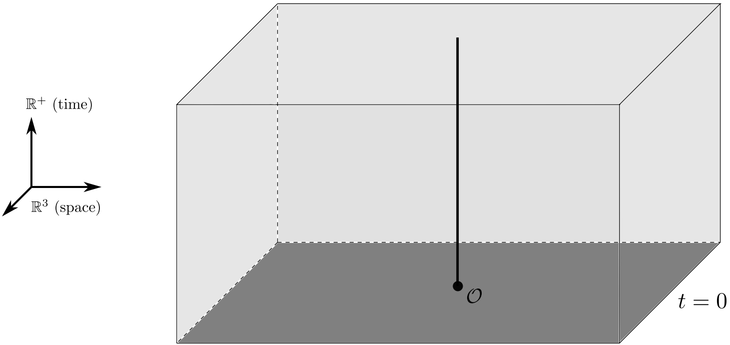
Before Einstein, *spacetime* (or *space and time*) was just a background arena on which physics took place, without any dynamics of its own. This all changed with general relativity: spacetime is no longer flat and boring, but curved in interesting and dramatic ways. What's more, it is dynamic, responding to the presence of matter and energy, as well as to its own previous state. But what exactly was "spacetime" before general relativity?

As a warm up, we'll review historical notions of spacetime. Our account will not be truly historical in some important respects:

- We'll use modern mathematical concepts and notation.
- We'll only give spacetime the *minimal* structure necessary to provide a foundation for the relevant physics.
- We sometimes lie for the sake of a clear mathematical narrative!

Aristotelian spacetime

- Spacetime is $\mathbb{R}^+ \times \mathbb{V}^3$, with time $t \in \mathbb{R}^+$ and points in space taking values in a 3-dimensional vector space $\mathbb{V}^3$.
- $\mathbb{V}^3$ is equipped with a positive-definite inner product (the "dot product").
- Choosing a basis for $\mathbb{V}^3$ that diagonalizes and normalizes the inner product (an *orthonormal basis*), $\mathbb{V}^3$ can be identified with $\mathbb{R}^3$ and the dot product is the standard dot product.
- There is a special time ($t = 0$), the "moment of creation", and a special place (the origin of $\mathbb{V}^3$), the "centre of the world".



Time goes up the page, and "space at a given time" is represented by a horizontal slice through spacetime. Space extends infinitely in all directions, while time extends to the future of (or "above") the plane $t = 0$. $\mathcal{O}$ is the origin of space and time, and the the origin of $\mathbb{V}^3$ at various times, i.e. the curve $(t, 0)$, is shown as a solid black straight line going up the page, emanating from $\mathcal{O}$.

Aristotelian relativity

Physical laws should "respect" the underlying structure of spacetime; they should be invariant under transformations which leave this structure invariant. The only such transformations are rotations of $\mathbb{V}^3$ about the origin – these leave both the origin and the dot product invariant.

Physical laws should be the same wherever we are on the globe!

Problems with Aristotelian spacetime

- Physical laws are not "protected" (by relativity) from being different at different altitudes or at different times – so why should they be be the same?
- Is the centre of the Earth (or even the Sun) really the centre of the Universe?

Atomist spacetime

A slightly more sophisticated try. Spacetime is $\mathbb{E} \times \mathbb{E}^3$, where $\mathbb{E}^n$ is the n-dimensional *affine space*. This can be thought of as a vector space with the origin removed: given any two points $p, q \in \mathbb{E}^n$, there is a corresponding vector $(p - q) \in \mathbb{V}^n$. As before, we equip the vector space with a positive-definite inner product.

There is no longer any special place (a "centre of the world"), nor is there a beginning of time – all times and all places are on an equal footing.

Particles at rest

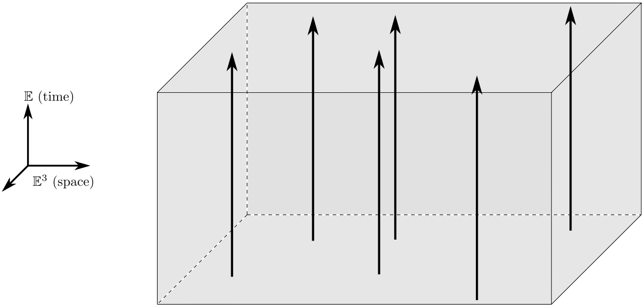
There are certain special paths through atomist spacetime. Given a curve

$$\begin{aligned}\gamma : \mathbb{R} &\to \mathbb{E} \times \mathbb{E}^3 \\ s &\mapsto (t(s), p(s)),\end{aligned}$$

we can construct a vector (depending on time) $v(s) \in \mathbb{V}^3$,

$$v(s) := p(s) - p(0).$$

For certain curves, $v(s) \equiv 0$. These are said to be the paths (or *worldlines* of "particles at rest".



A sketch of "Atomist" spacetime. Unlike "Aristotelian" spacetime, there is no special origin of time or space, both of which extend infinitely in all directions. The paths of several particles at rest are shown as solid black arrows.

Atomist relativity

- Again, physical laws should be invariant under transformations which leave the spacetime structure invariant.
- Now, these transformations are rotations (about *any* point) and translations, both in space and time.
- Physical laws are the same at all times, in all places, and don't depend on which direction we face!

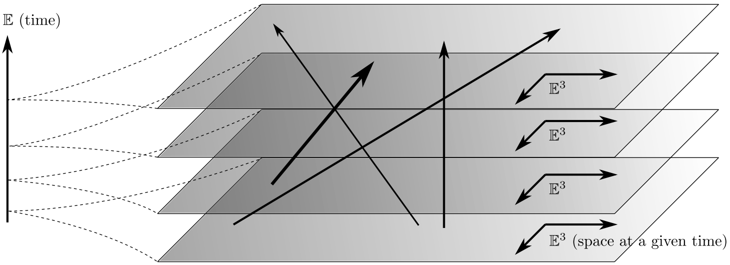
Problems with atomist spacetime

- Need a new account of gravity – can no longer say that objects "move towards the centre".
- Hard to account for projectile motion: why don't projectiles immediately change to being "at rest" once a force ceases to act on them?
- Major problems were identified by Galileo, who showed that you can't distinguish between particles "at rest" and those moving at a constant velocity.

Galilean spacetime

For those in the know, Galilean spacetime is a *fibre bundle* with base space $\mathbb{E}^1$ and fibre $\mathbb{E}^3$, where the fibres are equipped with the dot product as before, and the bundle is equipped with a connection.

Think of an $\mathbb{E}^1$ (a line), and, associated with each point on the line, there is an $\mathbb{E}^3$, which model *space at a given time*. Think of stacking the frames from a film on top of each other. There is no preferred way to line up the different $\mathbb{E}^3$s, but there are some special paths through spacetime – "straight lines". In addition to the usual rotations and translations, we are also free to *slide* the different $\mathbb{E}^3$s around, as long as we do so in a way which preserves all straight lines.



A sketch of Galilean spacetime. To each point in time we associate an $\mathbb{E}^3$, which corresponds to space at that time. Unlike in the previous examples, there is no preferred identification of the different $\mathbb{E}^3$'s, leading to the absence of *absolute space*. However, there is a special family of curves through spacetime, which are the *worldlines of inertial observers*, some of which are shown as arrows through spacetime in the figure

Galilean relativity

Easiest to state if we pick an (arbitrary) origin to space and time, so that we can use coordinates (t, x^i). Then physical laws should be invariant under *Galilean transformations*:

$$t \mapsto t + t_0$$
$$x^i \mapsto R^i{}_j x^j + v^i t + (x_0)^i,$$

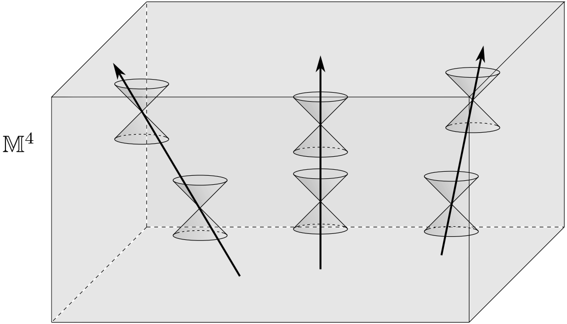
where R is an orthogonal matrix, v and x_0 are vectors and t_0 are constants.
Famously, Newton's laws are invariant under these transformations (though Newton, not knowing about fibre bundles, felt the need to introduce "absolute space", which was also absolutely undetectable!).

Problems with Galilean spacetime

- Main problem is the constancy of the speed of light – Galilean transformations can be used to set any velocity to zero!
- Although there is no "absolute space" (there is no sense in saying that some place, at time t_0 is "the same place" at a different time t_1), there is still an "absolute time": the only ambiguity in time is in the choice of an origin $t = 0$. But some of Einstein's thought experiments show that different observers will disagree on the order of various events (the *relativity of simultaneity*).

Minkowski spacetime

Space and time are no longer separate, but unified: spacetime is $\mathbb{M}^4$, a four-dimensional affine space. Instead of being equipped with a positive-definite, inner product, the associated vector space is equipped with an indefinite (but non-degenerate) quadratic form m with signature $(-,+,+,+)$.



A sketch of Minkowski spacetime $\mathbb{M}^4$. Space and time are no longer separate entities, but are merged into the single object spacetime. There is no longer a (unique) notion of space at a given time: horizontal slices are space at a given time *according to an inertial observer moving vertically up the page*, but other observers will split space and time differently. The *light cones* (or *null cones*) are invariant under Lorentz transformations, so all observers will agree on them - some of these are shown in the sketch. Worldlines of inertial observers are straight lines and (if the observers have nonzero mass) pass through the interior of the light cones.

Special relativity

This spacetime structure is invariant under the action of the *Poincaré group*, which consists of translations (in space and time) together with Lorentz transformations. More on this later.

Problems with Minkowski spacetime

How to incorporate gravity?