C7.5 Lecture 3: Newtonian gravity and the equivalence principle

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Newtonian gravity

The *gravitational potential* Φ is a function on spacetime solving Poisson's equation

$$\Delta\Phi = 4\pi G\rho,$$

where G is Newton's constant, ρ is a function from spacetime to $\mathbb{R}$ called the *matter density*, and Δ is the Laplacian operator: working in coordinates (x, y, z) where the inner product on $\mathbb{E}^3$ takes the standard form, we have

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2.$$

Particles moving in a gravitational field obey the equations of motion

$$m\ddot{x} = -m\nabla\Phi.$$

Theoretical problems with Newtonian gravity

All of this is fine on Aristotelian/Atomist/Galilean spacetimes, where "space at a given time" is $\mathbb{E}^3$, and we can solve Poisson's equation separately at every time. But on Minkowski space, there is no unique "space at a given time" – there are many different spacelike slices, corresponding to different inertial frames. Different choices lead to different gravitational potentials, and different predicted motions for particles moving in the gravitational field.

Another way to see this problem is to note that *the equations of Newtonian gravity are not invariant under Poincaré transformations*.

Empirical problems with Newtonian gravity

By the time Einstein began to work on General Relativity, some observations clashed with Newtonian gravity:

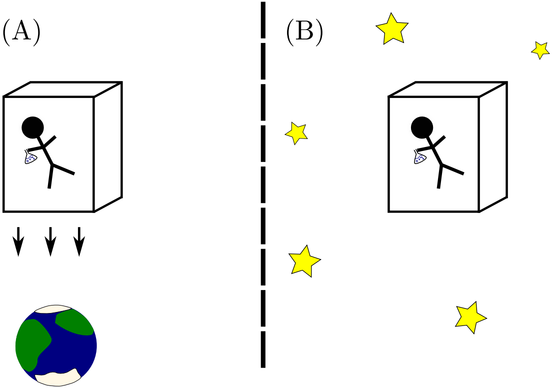
- The orbit of Mercury didn't match predictions (is there another planet, "Vulcan"?).
- Light is bent when it passes close to the sun, but Maxwell's equations don't include the gravitational potential. There is no easy way to include the potential either – Maxwell's equations are invariant under Poincaré transformations (but *not* Galilean transformations), while Newtonian gravity is invariant under Galilean transformations (but *not* Poincaré transformations).

Philosophical problems with Newtonian gravity: the equivalence principle

Two situations:
(A) you are in a closed box (e.g. an elevator) in free-fall towards the Earth, and
(B) you are in the same box, but floating deep in space.

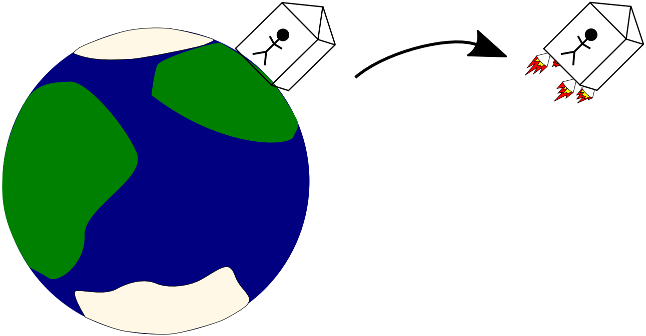
How to tell them apart? What experiment should you perform to see if you are in situation (A) or (B)? Within Newtonian mechanics, there is no *local*(i.e. on a sufficiently small length scale and time scale) experiment you can perform that does the job.

The reason for this is that the gravitational force is proportional to the mass of the object it acts upon – just like “fictitious forces” (e.g. centrifugal force).



Einstein's thought experiment: (A) a scientist in an elevator falling to Earth, while in (B) they are floating in space. No local experiment can distinguish between (A) and (B).

An amusing variant of this thought experiment:



Did this just happen?

If these two situations can't be distinguished empirically, why do we describe them so differently: in situation (A) you and all your experimental apparatus is acted upon by the force of gravity, in situation (B) it is not.

Three versions of the equivalence principle

The weak equivalence principle: the motion of test particles moving in a gravitational field depends only on their initial positions and velocities.

The Einstein equivalence principle: all local, non-gravitational experiments performed in freely-falling laboratories will obtain the same results.

The strong equivalence principle: the motion of sufficiently small bodies moving in a gravitational field depends only on their initial positions and velocities. Also, all local experiments performed in freely-falling laboratories will obtain the same results.

Hints of spacetime curvature

In pre-GR spacetimes, particles/observers experiencing no external force moved along "straight lines". The equivalence principle suggests that observers who experience no external forces are *freely falling*, so the paths of these observers should play the role of "straight lines". However, in the presence of a gravitational field, these paths are not straight – for example, they can orbit the Earth.

We can still consider these paths to be "straight" in a sense (i.e. *length minimizing*[1]), but only if spacetime itself is curved!

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[1] Actually, *proper-time* maximising.