C7.5 Special Lecture 4.5 (examples)

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Example 1 (tensor fields)

Show that, if T is a rank $(0,2)$ tensor field, X, Y and Z are vector fields, and a, b and c are scalar fields, then

$$T(aX + bY, cZ) = acT(X, Z) + bcT(Y, Z).$$

Note that a, b and c don't have to be constants – they are scalar fields, and can vary from point to point in spacetime.

In index notation, this reads

$$T_{ab}(aX^a + bY^a)cZ^b = acT_{ab}X^aZ^b + bcT_{ab}Y^aZ^b,$$

so this question legitimises some manipulation which is natural in index notation.

Choose an arbitrary point $p \in \mathbb{M}^4$. Then $T\big|_p$ is a rank $(0,2)$ tensor at p, $X\big|_p$, $Y\big|_p$ and $Z\big|_p$ are vectors at p and $a\big|_p$, $b\big|_p$ and $c\big|_p$ are constants (they are the values of the scalar fields a, b and c at the point p).
Hence we can calculate

$$T(aX + bY, cZ)\big|_p = T\big|_p \left(a\big|_p X\big|_p + b\big|_p Y\big|_p, c\big|_p Z\big|_p\right)$$

$$= a\big|_p c\big|_p T\big|_p \left(X\big|_p, Z\big|_p\right) + b\big|_p c\big|_p T\big|_p \left(Y\big|_p, Z\big|_p\right)$$

(by the linearity of $T\big|_p$)

$$= \left(acT(X,Z) + bcT(Y,Z)\right)\big|_p.$$

Since p was arbitrary, this holds at all points, i.e.

$$T(aX + bY, cZ) = acT(X,Z) + bcT(Y,Z).$$

Example 2 (proper time along an accelerated worldline)

In this question we treat the Earth as being at rest in a system of inertial coordinates.

An astronaut begins at rest on the Earth, and plans to visit a distant planet, which is at rest relative to the Earth a proper distance D away. Her spaceship accelerates during the journey at a constant rate α, i.e.

$$m_{ab}a^a a^b = \alpha^2,$$

where a is the four-acceleration of the astronaut.

What is the path of the astronaut, in terms of inertial coordinates centred on the Earth? How much time passes, from the astronaut's point of view, until she reaches her destination?

We can choose coordinates (t, x, y, z), where the worldline of the Earth is simply $(t, 0, 0, 0)$, and the worldline of the distant planet is given by $(t, D, 0, 0)$. Then the worldline of the astronaut is

$$(t(\tau), x(\tau), 0, 0)\,.$$

Since τ is the proper time along the astronaut's worldline, we have

$$-\dot{t}^2 + \dot{x}^2 = -1,$$

where 'dots' are derivatives w.r.t. τ. The acceleration of the astronaut is

$$a = \begin{pmatrix} \ddot{t} \\ \ddot{x} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\dot{x}\ddot{x}}{\sqrt{1+\dot{x}^2}} \\ \ddot{x} \\ 0 \\ 0 \end{pmatrix}$$

Since $m_{ab}a^a a^b = \alpha^2$, we have

$$\alpha^2 = \frac{\ddot{x}^2}{1+\dot{x}^2}.$$

Moreover, $\dot{x} \geq 0$. Solving this equation for $\dot{x}(\tau)$, we find that

$$\dot{x} = \sinh(\alpha\tau + c),$$

for some constant c. But since $\dot{x}(0) = 0$, we have $c = 0$. Integrating again, and using $x(0) = 0$, we find that

$$x(\tau) = \frac{1}{\alpha}\left(\cosh(\alpha\tau) - 1\right).$$

In view of $\dot{t} = \sqrt{1+\dot{x}^2}$, and $t(0) = 0$, we find that

$$t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau)$$

If we like, we can write the path of the astronaut in terms of inertial coordinates as

$$\left(t, \frac{1}{\alpha}\left(\sqrt{\alpha^2 t^2 + 1} - 1\right), 0, 0\right).$$

The time that passes for the astronaut is the proper time along her worldline, i.e. τ. She reaches the distant planet when

$$\tau = \frac{1}{\alpha}\cosh^{-1}(1 + \alpha D).$$

Note that, if $\alpha D \gg 1$, then $\tau \sim \frac{1}{\alpha}\log(\alpha D)$. So, no matter how large D is, if α is sufficiently large, then the astronaut can reach the planet after a "reasonable" proper time. On the other hand, when the she reaches the planet,

$$t = \sqrt{D^2 + 2\frac{D}{\alpha}},$$

so, no matter how large α is, it takes at least a time D to reach the planet, as viewed from Earth.

Example 3 (null curves which are not straight lines)

Give an example of a null curve in Minkowski space which is not a straight line in inertial coordinates.

Consider the curve given, in inertial coordinates, by

$$(t, x, y, z) = (\lambda, \sin\lambda, \cos\lambda, 0).$$

Then the tangent vector to this curve (parametrised by λ) is

$$v = \begin{pmatrix} 1 \\ \cos\lambda \\ -\sin\lambda \\ 0 \end{pmatrix},$$

and we can compute its Minkowski norm $m_{ab}v^a v^b$, which is $-1 + \cos^2\lambda + \sin^2\lambda = 0$. Hence this curve is null (but it is clealy not a straight line!).

Example 4 (Ladders and barns)

This is a classic thought experiment. We'll approach the solution more geometrically, rather than just through Lorentz transformations.

I stand outside a barn and watch Bob runs very quickly (at some significant fraction of the speed of light!) at a constant speed in a straight line through the barn, carrying a ladder. The barn has doors at both the front and back, and two of my friends stand near the doors ready to close them.

The proper length of the ladder is ℓ, while the proper length of the barn is b, with $b < \ell$. Show that, if Bob runs fast enough, my friends can both (temporarily) close the doors with Bob and the ladder inside. They then open the doors again so that Bob and the ladder can pass through the barn.

What happens from Bob's point of view?

We can work in inertial coordinates where the barn is at rest – these will correspond to “what I see”. In these coordinates, the worldline corresponding to the front of the barn is at $(t, x, y, z) = (\lambda, 0, 0, 0)$, while that correponding to the back of the barn is at $(\lambda, b, 0, 0)$.

The worldline of the front of the ladder is $(\lambda, v\lambda, 0, 0)$, where v is the relative velocity of the Bob (we can choose the coordinates so that the ladder enters the barn at $t = \lambda = 0$), and the back of the ladder follows the worldline $(\lambda, v\lambda - a, 0, 0)$ for some constant a (which is *not* ℓ!).

If $a \neq \ell$, what is the constant a, and how do we find it? One way is to use a Lorentz transformation to switch to the rest frame of the ladder – if we use inertial coordinates where the ladder is at rest, then the proper length of the ladder will be the same as its coordinate length.

An alternative approach – and the one we will use – is to stay in our chosen coordinate system, and to calculate the proper length of the curve which joins the front of the ladder to the back of the ladder which is orthogonal (in the Minkowski sense) to the worldlines of the points making up the ladder.

There are many equivalent definitions of the length of a rigid body: a third one is half the *proper time* along the worldline at one end of the rigid body between the emission and reception of a light signal which bounces off the other end of the body.

The worldlines of the points making up the ladder are given by $(t, x, y, z) = (\lambda, v\lambda - c, 0, 0)$, where $c \in [0, a]$: their tangent vectors are given by

$$\begin{pmatrix} 1 \\ v \\ 0 \\ 0 \end{pmatrix}.$$

Hence, a spacelike straight line orthogonal to these worldline is given by $(-v\lambda, -\lambda, 0, 0)$. This curve meets the front of the ladder at $\lambda = 0$, and the back of the ladder at $\lambda = \frac{a}{1-v^2}$.

We want to calculate the proper length of the curve $(-v\lambda, -\lambda, 0, 0)$, with $\lambda \in [0, \frac{a}{1-v^2}]$. λ is not the proper distance along this curve: its tangent vector X has norm $m(X, X) = 1 - v^2$. Parametrising this curve by proper length s instead of λ, we find that

$$s = \lambda\sqrt{1 - v^2}.$$

Hence when $\lambda = \frac{a}{1-v^2}$, $s = \frac{a}{\sqrt{1-v^2}}$, and this is the proper length of the ladder, ℓ. In other words,

$$a = \ell\sqrt{1 - v^2}.$$

Hence we see that, if

$$v \geq \sqrt{1 - \frac{b^2}{\ell^2}},$$

then, from my point of view, the entire ladder can fit in the barn.

Both doors of the barn can be closed (with the ladder inside) for

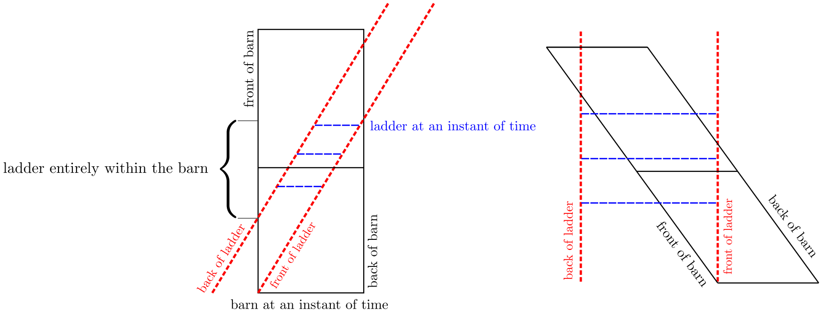
$$\frac{\ell}{v}\sqrt{1-v^2} \leq t \leq \frac{b}{v}.$$

To find out how things look from Bob's point of view, we can do a Lorentz transformation: coordinates in which Bob is at rest are given by (t', x', y', z'), where

$$(t', x', y', z') = (\gamma t - \gamma v x, -\gamma v t + \gamma x, y, z),$$

with $\gamma = \frac{1}{\sqrt{1-v^2}}$.

In Bob's coordinates, the front of the barn follows the worldline $(\lambda, -v\lambda, 0, 0)$, and the back of the barn follows the worldline $(\lambda, -v\lambda + b\sqrt{1-v^2}, 0, 0)$. The front door is closed for $\frac{\ell}{v} \leq t' \leq \frac{b\gamma}{v}$, while the back door is closed for $\frac{\ell}{v} - \gamma v b \leq t' \leq \frac{b\gamma}{v} - \gamma v b$: there is always at least one door open, and the entire ladder is never all in the barn at one time!



On the left we see the situation from the point of view of an inertial observer at rest relative to the barn. There is a period of time where, for this observer, the entire ladder appears to be within the barn. On the right we see things from Bob's point of view, in which the ladder is at rest: now, the ladder never appears to fit within the barn.