C7.5 Lecture 5: Differential geometry 1

Manifolds and coordinate charts

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We have two goals in investigating curved spacetimes:

1. We need to reconstruct all of the mathematical tools we needed to do physics in Minkowski spacetime: curves, tangent vectors, proper time and distance, tensor fields etc. Importantly, we also want to be able to do calculus using these objects (vector calculus etc.) – this will turn out to be our hardest job.
2. We need to understand the new structure that we get in a curved spacetime – namely, curvature – and relate this, somehow, to gravity.

Manifolds and coordinate charts

The basic object in differential geometry – and our model for a curved spacetime – is a *manifold*. A manifold $\mathcal{M}$ is a topological space[1] where sufficiently small open sets "look like" open sets in $\mathbb{R}^n$.

We make this precise as follows: $\forall p \in \mathcal{M}$, $\exists$ an open neighbourhood U of p and a map $\phi_U : U \to \mathbb{R}^n$.

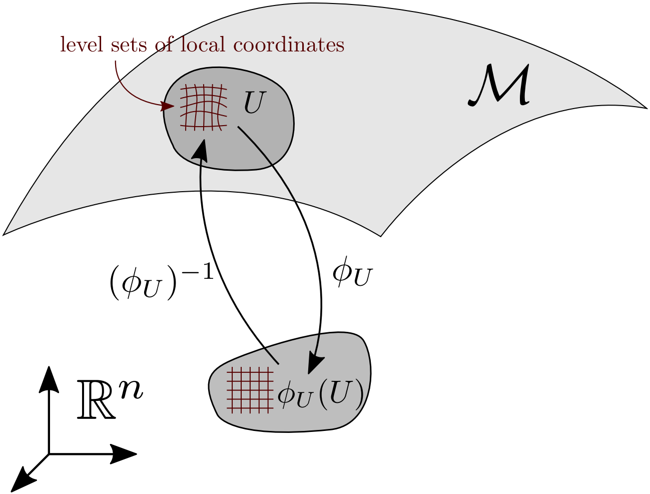
- ϕ_U is a *chart* or *coordinate chart*.
- U is a *coordinate patch*.
- ϕ_U is a bijection between U and $\phi_U(U)$.
- Both ϕ_U and ϕ_U^{-1} are continuous.
- $n \in \mathbb{N}$ is the *dimension of the manifold*.

[1] It is also required to be *second countable* and *Hausdorf*, but these technical details will not concern us.

We can use the chart ϕ_U to define *local coordinates* x^a in the set U. These are defined as the 'pull-back' of the standard coordinates on $\mathbb{R}^n$: in a slight abuse of notation, for each $a \in \{0, 1, \ldots, n-1\}$ we set

$$x^a(p) = x^a(\phi_U(p)),$$

where, on the right hand side, $x^a(\phi_U(p))$ is just the value of the standard coordinate x^a in $\mathbb{R}^n$ at the point $\phi_U(p)$.

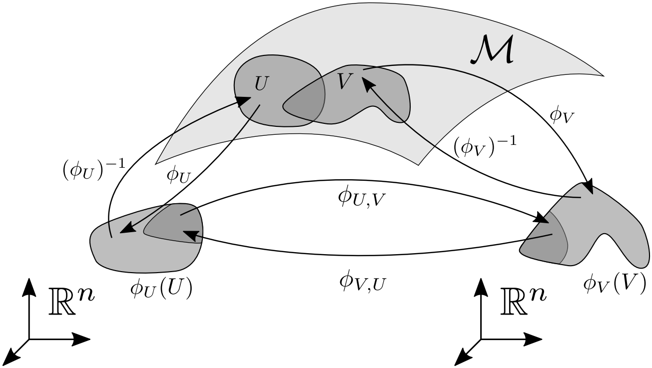


A manifold $\mathcal{M}$, with a coordinate patch U and a chart ϕ_U. Local coordinates are defined in the patch ϕ_U by using the usual coordinates on $\mathbb{R}^n$ and the chart ϕ_U.

Generally, we need more than one chart to cover the manifold $\mathcal{M}$. An *atlas* is a collection of charts covering the entire manifold. It can happen that two charts overlap - that is, we can have charts ϕ_U and ϕ_V with $U \cap V \neq \emptyset$. On the overlap, we define *transition functions*:

$$\begin{aligned} \phi_{U,V} : \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto \phi_V \circ (\phi_U^{-1})(x) \end{aligned}$$

(see figure 2).



Here the coordinate patches U and V overlap, allowing us to define the transition functions $\phi_{U,V}$ and $\phi_{V,U}$. For a smooth manifold, these transition functions are smooth.

The transition functions are maps from some open set of $\mathbb{R}^n$ to another open set of $\mathbb{R}^n$.

Hence we can make sense of (for example) the differentiability of these maps. We will always work with *smooth manifolds*, meaning that we always use atlases in which all transition functions are C^∞.

Some examples of manifolds:

1. any (finite dimensional) vector space,
2. the n-sphere $\mathbb{S}^n$,
3. a cone without the vertex point,
4. the torus $\mathbb{S} \times \mathbb{S}$, etc.