C7.5 Lecture 6: Differential geometry 2

Curves, vectors, and tensors

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Curves and tangent vectors

As before, we define a curve

$$\gamma : [0, 1] \text{ (or } \mathbb{R} \text{ (or } \mathbb{R}^+) \to \mathcal{M}.$$

What about a tangent vector? Our manifold does not have an affine space structure, nor are there special sets of inertial coordinates we can use.

We *can* still differentiate functions along a curve: given $f : \mathcal{M} \to \mathbb{R}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f \circ \gamma := V(f),$$

where we use this equation to *define* the "vector" V. It is an operator which acts on scalar fields f via the above formula.

Vectors satisfy the following two important properties: for constants a, $b \in \mathbb{R}$ and functions f, $g : \mathcal{M} \to \mathbb{R}$

1. *Linearity*:

$$V(af + bg) = aV(f) + bV(g).$$

2. *The Leibniz rule*:

$$V(fg) = gV(f) + fV(g).$$

Tangent vectors in local coordinates

In terms of local coordinates x^a, we can set

$$V(f)\big|_p = V(f \circ \phi_U^{-1} \circ \phi_U)\big|_p$$
$$= V\left(\tilde{f}(x^a)\right)\big|_p,$$

where $\tilde{f} = f \circ \phi_U^{-1}$. Note that $\tilde{f} : \mathbb{R}^n \supset U \to \mathbb{R}$, and $x^a = \phi_U(p)$.

Using the chain rule we have

$$V(f)\big|_p = V(x^a)\big|_p \frac{\partial \tilde{f}}{\partial x^a}\bigg|_{x^a(p)}$$
$$= V^a \partial_a \tilde{f}.$$

Since this formula holds in *all* local coordinates, we write $V = V^\mu \partial_\mu$. By a common abuse of notation, people often write f for $\tilde{f} = f \circ \phi_U^{-1}$, although these are two different objects: f is a function on the manifold, while $\tilde{f}$ is a function of the local coordinates x^a (of course, they take the same value at corresponding points!).

Vectors and the tangent space

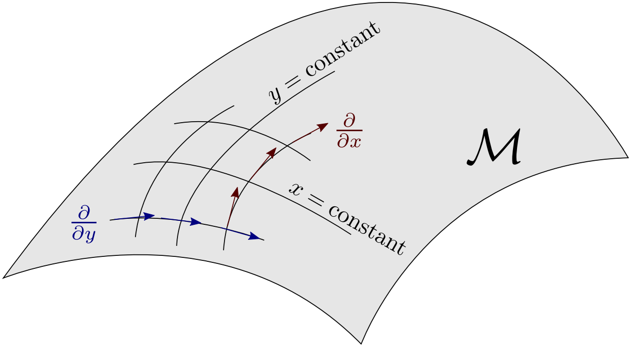
A *vector* at a point $p \in \mathcal{M}$ is the tangent vector to some curve[1] through p, at the point p.

The *tangent space at p*, $T_p(\mathcal{M})$ is the set of all vectors at p.

[1]Strictly speaking we need to talk about equivalence classes, because there are multiple curves with the same tangent vector. Two curves γ and γ', with tangent vectors V and V' at p are said to define the same vector if $V(f) = V'(f)$ for all f.

$T_p(\mathcal{M})$ is a vector space with the same dimension as the dimension of the manifold.

Given some local coordinates x^a, we can define the vectors $\partial_a = \frac{\partial}{\partial x^a}$ as the vectors tangent to the curves along which x^a changes while x^b, $b \neq a$ remain constant, parametrised by x^a (see figure 1). Such vector fields are sometimes called *coordinate induced vector fields*.



The *coordinate induced vector field* $\frac{\partial}{\partial x}$ points in the direction where x changes while all the other coordinates (here, the coordinate y) remain the same. Similarly, $\frac{\partial}{\partial y}$ points in the direction where y changes while x remains constant.

Covectors and tensors

The *cotangent space* $T_p^*(\mathcal{M})$ is the dual space of the vector space $T_p(\mathcal{M})$, i.e. it consists of all linear maps (called *covectors*) from the tangent space to $\mathbb{R}$.

A *tensor of rank* (n, m) is an element of $(T_p(\mathcal{M}))^n \times \left(T_p^*(\mathcal{M})\right)^m$. Equivalently, it is a multi-linear map from $(T_p(\mathcal{M}))^m \times \left(T_p^*(\mathcal{M})\right)^n$ to $\mathbb{R}$.

Tensor components and their transformation laws

Given local coordinates x^a, the components of the vector X are

$$X^a := X(x^a).$$

By the chain rule, for any scalar function $f : \mathcal{M} \to \mathbb{R}$ we have

$$X(f) = X(f \circ \phi_U^{-1} \circ \phi_U) = X(x^a) \frac{\partial}{\partial x^a}(f \circ \phi_U^{-1}) = X^a \frac{\partial}{\partial x^a}(f \circ \phi_U^{-1}).$$

Note that $\phi_U(p) = \big(x^0(p), \ldots x^{n-1}(p)\big)$, and $f \circ \phi_U^{-1} : \mathbb{R}^n \to \mathbb{R}$.

In particular, the components of the vector ∂_b are

$$(\partial_b)^a = \partial_b(x^a) = \delta_b^a,$$

so

$$(\partial_b)(f) = \frac{\partial}{\partial x^b}(f \circ \phi_U^{-1}).$$

Now suppose we change coordinates in a neighbourhood of the point p, from the coordinates x^a to coordinates $y^{a'}(x^a)$. Then the new components of the vector X are, using the chain rule,

$$(X')^{a'} = X(y^{a'}) = \frac{\partial y^{a'}}{\partial x^a} X(x^a) = \frac{\partial y^{a'}}{\partial x^a} X^a.$$

This is the *transformation law for vectors*.

Let η be a covector. Then the components of η are defined to be

$$\eta_a := \eta(\partial_a).$$

Note that

$$\eta(X) = \eta(X^a \partial_a) = X^a \eta_a$$

Since this holds in *any* coordinate system, we write $\eta(X) = X^\mu \eta_\mu = \eta_\mu X^\mu$.

Under a change of coordinates as before, we have

$$\eta(X) = \eta_a X^a = (\eta')_{a'} (X')^{a'} = (\eta')_{a'} \frac{\partial y^{a'}}{\partial x^a} X^a,$$

so we must have

$$\begin{aligned} \frac{\partial y^{a'}}{\partial x^a} (\eta')_{a'} &= \eta_a \\ \Rightarrow (\eta')_{a'} &= \frac{\partial x^a}{\partial y^{a'}} \eta_a \end{aligned}$$

using the inverse function theorem. This is the *covector transformation law*.

Now, under a change of coordinates as above, we have

$$\eta(X) = \eta_a X^a = (\eta')_{a'} X^{a'} = (\eta')_{a'} \frac{\partial y^{a'}}{\partial x^a} X^a$$

so we must have

$$\frac{\partial y^{a'}}{\partial x^a} (\eta')_{a'} = \eta_a$$
$$\Rightarrow (\eta')_{a'} = \frac{\partial x^a}{\partial y^{a'}} \eta_a$$

using the inverse function theorem. This is the *covector transformation law*.

More general tensors transform in the "obvious" way – as if they were the product of a bunch of vector and covector fields :

$$(T')^{a_1' a_2' \ldots a_n'}{}_{b_1' b_2' \ldots b_m'} = \frac{\partial y^{a_1'}}{\partial x^{a_1}} \frac{\partial y^{a_2'}}{\partial x^{a_2}} \cdots \frac{\partial y^{a_n'}}{\partial x^{a_n}} \frac{\partial x^{b_1}}{\partial y^{b_1'}} \frac{\partial x^{b_2}}{\partial y^{b_2'}} \cdots \frac{\partial x^{b_m}}{\partial y^{b_m'}} T^{a_1 a_2 \ldots a_n}{}_{b_1 b_2 \ldots b_m}.$$