C7.5 Lecture 7: Differential geometry 3

Tensor fields and the metric

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Tensor fields

A *tensor field* is an assignment of a tensor to all points in spacetime (or, occasionally, some open subset of points).

We always work with smooth (C^∞) tensor fields. To check the differentiability of a tensor field we can simply examine its components in a chart. Since the transition functions are smooth, this is a coordinate-independent notion in a smooth manifold.

We can also consider a rank (n, m) tensor field F as a linear operator at each point p,

$$F\Big|_p : \left(T_p^*(\mathcal{M})\right)^m \times \left(T_p(\mathcal{M})\right)^n \to \mathbb{R}.$$

This means that it is C^∞-*linear* in its arguments. For example, a covector field η is a function from vector fields to the reals, satisfying

$$\eta(aX + bY) = a\eta(X) + b\eta(Y)$$

for all scalar fields a, b and all vector fields X, Y.

Note that a and b are allowed to vary (smoothly) from point to point – they do not have to be constant!

The *tangent bundle* $T(\mathcal{M})$ is the union of all of the tangent spaces of the manifold:

$$T(\mathcal{M}) = \bigcup_{p \in \mathcal{M}} T_p(\mathcal{M})$$

An element of the tangent bundle is a pair (p, X), where p is a point in the manifold and X is a vector at p. In an exactly analogous way, we can define the *cotangent bundle* as the union of all the cotangent spaces.

Why is it a "bundle", and not, for example, a vector space? There is no way to add two elements of the tangent bundle (p, X) and (q, Y), unless $p = q$. Although vectors at p and q are both elements of n-dimensional vector spaces ($T_p(\mathcal{M})$ and $T_q(\mathcal{M})$ respectively), there is nothing "connecting" the different tangent spaces at different points – there is no vector in one tangent space which "corresponds" to a given vector in another tangent space.

Examples of tensor fields

In a coordinate patch U, we can define the vector fields $\partial_a = \frac{\partial}{\partial x^a}$ as above.

Given a smooth function $f : \mathcal{M} \to \mathbb{R}$ (a *scalar field*), we can define the covector field $\mathrm{d}f$ by its action on an arbitrary vector field X:

$$\mathrm{d}f(X) := X(f).$$

Since, for all p, this defines a linear map from $T_p(\mathcal{M}) \to \mathbb{R}$, this defines a covector field (easy to check that $\mathrm{d}f(aX + bY) = a\mathrm{d}f(X) + b\mathrm{d}f(Y)$).

Coordinate induced covector fields

In a coordinate patch $\mathcal{U}$, the coordinate functions x^a are themselves smooth functions. Hence we can construct the coordinate differentials: the covector fields $\mathrm{d}x^a$.

These covectors form a basis for the cotangent space at any point, which is in fact the dual basis to the basis of coordinate induced vector fields ∂_a, i.e.

$$\mathrm{d}x^a(\partial_b) = \delta^a_b.$$

We can also expand any covector in terms of this basis (**exercise**):

$$\eta = \eta_a \mathrm{d}x^a \qquad \text{where } \eta_a = \eta(\partial_a).$$

As before, we can define the *Kronecker delta*, which is a $(1,1)$ tensor field, defined by its action on an arbitrary vector field X and covector field η:

$$\delta(X,\eta) = \eta(X)$$

This defines a linear map from $T_p(\mathcal{M}) \times T_p^*(\mathcal{M}) \to \mathbb{R}$, so δ is a tensor field.

Forming new tensors out of old

There are many ways to form new tensors out of old ones.

Given two vector fields X and Y, we can form their product XY, which is a rank $(2,0)$ tensor field with components $(XY)^{ab} = X^a Y^b$ in any coordinate system. Alternatively, we can define it by its action on a pair of covectors *eta*, μ:

$$(XY)(\eta, \mu) = \eta(X)\mu(Y).$$

Given a $(1,1)$ tensor field T, we can form a scalar field by contracting its indices: we can form the scalar field $T^\mu{}_\mu$, whose value in any coordinate system is $T^a{}_a$.

For a coordinate-free definition of $T^\mu{}_\mu$ at some point p, we introduce a basis (e_a) for $T_p(\mathcal{M})$, and the dual basis for $T_p^*(\mathcal{M})$, (f^a), where $f^a(e_b) = \delta^a_b$. Then

$$T^\mu{}_\mu\big|_p = \sum T(f_a, e^a).$$

Check that these definitions are independent of the choice of local coordinates (for $T^a{}_a$)) or of the basis e_a (in the coordinate independent definition).

If $T_{\mu\nu}$ is a rank $(0,2)$ tensor field, then we can define its symmetric and antisymmetric parts, which are the tensors $T_{(\mu\nu)}$ and $T_{[\mu\nu]}$ with components

$$T_{(ab)} := \frac{1}{2}\left(T_{ab} + T_{ba}\right)$$
$$T_{[ab]} := \frac{1}{2}\left(T_{ab} - T_{ba}\right).$$

There are many, many other ways to combine tensors, contract indices etc. etc. to form new tensors.

The metric tensor

Finally we introduce the metric tensor g. Manifolds equipped with a metric tensor are called *Lorentzian manifolds* (if the metric has signature $(-,+,+,+)$. If the metric had signature $(+,+,+,+)$, then we would call it a *Riemannian manifold* instead).

The metric is a symmetric, rank $(0,2)$ tensor field. It generalises the Minkowski metric m to a manifold.

For a vector field X, we define the covector field $X^\flat$ by

$$X^\flat(Y) := g(X, Y) \quad \text{for all vector fields } Y$$

The metric is *non-degenerate*: $X^\flat = 0$ if and only if $X = 0$. In components,

$$(X^\flat)_a = g_{ab} X^b = X_a.$$

The metric g has signature $(-,+,+,+)$. This means that, in any coordinate system, at any point in the manifold, the matrix $g_{ab} = g(\partial_a, \partial_b)$ has signature $(-,+,+,+)$ (i.e. one negative and three positive eigenvalues). This is also a basis-independent notion.

The metric g plays the same role as the Minkowski metric m did in special relativity:

- A nonzero vector X is *timelike* if $g(X, X) < 0$, spacelike if $g(X, X) > 0$ and null if $g(X, X) = 0$.
- Curves are timelike/spacelike/null if their tangent vector is everywhere timelike/spacelike/null.
- On a timelike curve we define the proper time as the parameter such that the tangent vector V satisfies $g(V, V) = -1$. Similarly, on a spacelike curve the proper distance is defined so that $g(V, V) = 1$.

Notation for the metric

In terms of local coordinates x^a, we write

$$g = g_{ab} \mathrm{d}x^a \mathrm{d}x^b = \mathrm{d}s^2$$

Often we take for granted that the metric is symmetric, and so for brevity we write a non-symmetric expression, with the understanding that the metric is found by symmetrising. For example, we might write

$$g = \mathrm{d}x^1 \mathrm{d}x^2,$$

which should be understood as

$$g = \frac{1}{2} \mathrm{d}x^1 \mathrm{d}x^2 + \frac{1}{2} \mathrm{d}x^2 \mathrm{d}x^1,$$

i.e. $g_{12} = g_{21} = \frac{1}{2}$.

The quantity $\mathrm{d}s^2$, which is really just the metric tensor, is sometimes called the *line element*.

We also define the *inverse metric* g^{-1}, a rank $(2,0)$ tensor defined by

$$g^{-1}(X^\flat, \eta) = \eta(X)$$

for all vector fields X and covector fields η. In components, this reads

$$(g^{-1})^{ab} X_a \eta_b = (g^{-1})^{ab} g_{ac} X^c \eta_b = X^a \eta_a = \delta^b_c X^c \eta_b.$$

Since this holds for all X and η, it follows that $(g^{-1})^{ab} g_{ac} = \delta^b_c$, i.e. the matrix $(g^{-1})^{ab}$ is the inverse of the matrix g_{ab}.

We can use the inverse metric to raise indices: for a covector field η, define the vector field $\eta^\sharp$ by

$$g(\eta^\sharp, Y) = \eta(Y)$$

for all vector fields Y. In components

$$(\eta^\sharp)^a = (g^{-1})^{ab}\eta_b = \eta^a$$

A common notation is to avoid including the inverse sign when writing the inverse metric components, i.e.

$$(g^{-1})^{ab} = g^{ab}.$$

This is actually consistent with our notation: raising both indices on the metric:

$$g^{ab} = (g^{-1})^{ac}(g^{-1})^{bd}g_{cd} = (g^{-1})^{ac}\delta^b_c = (g^{-1})^{ab}.$$