C7.5 Lecture 8: Differential geometry 4

Calculus on manifolds

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We are still missing one key tool, before we can start to do physics on curved spaces: the ability to do calculus. It's possible to develop a theory of both differentiation and integration on manifolds, but in this course we will focus on differentiation. This will allow us to write down physical laws and solve equations of motion, and also eventually to understand curvature.

What do we already know?

We know how to differentiate scalar fields: given a scalar field f, we defined the covector field $\mathrm{d}f$ by

$$\mathrm{d}f(X) = X(f).$$

This is the generalization of the 'gradient of a function' to manifolds, and $X(f)$ is the 'directional derivative' of f in the direction of the vector X.

The components $\mathrm{d}f$ in the coordinate system x^a are given by

$$(\mathrm{d}f)_a = (\mathrm{d}f)(\partial_a) = \partial_a(f).$$

Since this holds in *any* coordinate system, we write

$$(\mathrm{d}f)_\mu = \partial_\mu f.$$

What about differentiating vector fields? There is one obvious idea: choose some local coordinates and differentiate the components of the vector field, and form the $(1,1)$ tensor field with those components. Unfortunately this won't work: consider a change of coordinates $x^a \to y^{a'}$. Then

$$\begin{aligned} \partial_a X^b &= \frac{\partial y^{a'}}{\partial x^a} \partial'_{a'} \left(\frac{\partial x^b}{\partial y^{b'}} (X')^{b'} \right) \\ &= \frac{\partial y^{a'}}{\partial x^a} \frac{\partial x^b}{\partial y^{b'}} \partial'_{a'} (X')^{b'} + \frac{\partial y^{a'}}{\partial x^a} \left(\frac{\partial^2 x^b}{\partial y^{a'} \partial y^{b'}} \right) (X')^{b'}. \end{aligned}$$

From this we see that the expression $\partial_a X^b$ *doesn't* transform like the components of a $(1,1)$ tensor field – in other words, the definition above depends on which coordinate system we start in.

What if we try using index-free notation? Suppose that we want to differentiate the vector field X in the direction of the vector field Y. We could try to define a vector field $Y(X)$ which acts on scalar fields f as

$$Y(X)(f) := Y(X(f)).$$

Although this obeys linearity, it does not obey the Leibniz rule, and so this does not define a vector field. In other words, there is no curve with tangent vector V, such that $V(f) = Y(X)(f)$ for all smooth functions f.

The reason this can't work is fairly obvious: the expression above depends on the second derivatives of f, whereas a vector field only takes first derivatives of f.

Affine connections

There are three approaches to taking derivatives vector fields and other higher-order tensor fields:

1. the exterior derivative (see problem sheet 2, only works for completely antisymmetric $(0, n)$ tensor fields),
2. the Lie derivative (see GR2), and
3. affine connections.

We'll work with affine connections. Our plan is:

1. Define affine connections in the abstract and deduce some of their properties.
2. Show that many different affine connections exist on a smooth manifold.
3. Show that, on Lorentzian manifold, there is a unique preferred connection with extra special properties.

An affine connection is a map from a pair of vector fields to a vector field

$$\Gamma : (X, Y) \mapsto \nabla_X Y$$

with the following properties:

1. ∇ is C^∞-*linear* in the first variable: for all scalar fields f and vector fields X, Y, Z,

$$\nabla_{fX+Y} Z = f \nabla_X Z + \nabla_Y Z.$$

2. ∇ satisfies the Leibniz rule in the second variable:

$$\nabla_X (fY + Z) = f \nabla_X Y + (X(f))Y + \nabla_X Z.$$

Components of a connection

We can define the *components of a connection*, also called the *Christoffel symbols*. These are defined, with respect to the local coordinates x^a, as follows:

$$\nabla_{\partial_b}\partial_c := \Gamma^a_{bc}\partial_a$$
$$\Leftrightarrow \Gamma^a_{bc} = (g^{-1})^{ad} g(\nabla_{\partial_b}\partial_c\,,\,\partial_d)$$

We will usually write ∇_a instead of ∇_{∂_a}.

The Christoffel symbols Γ^a_{bc} are *not* the components of a $(1,2)$ tensor field. If we change coordinates, then

$$\begin{aligned}
(\Gamma')^{a'}_{b'c'} &= \mathrm{d}y^{a'}\left(\nabla_{b'}\partial'_{c'}\right) \\
&= \frac{\partial y^{a'}}{\partial x^a}\mathrm{d}x^a\left(\nabla_{b'}\partial'_{c'}\right) \\
&= \frac{\partial y^{a'}}{\partial x^a}\mathrm{d}x^a\left(\nabla_{\frac{\partial x^b}{\partial y^{b'}}\partial_b}\left(\frac{\partial x^c}{\partial y^{c'}}\partial_c\right)\right) \\
&= \frac{\partial y^{a'}}{\partial x^a}\mathrm{d}x^a\left(\frac{\partial x^b}{\partial y^{b'}}\nabla_{\partial_b}\left(\frac{\partial x^c}{\partial y^{c'}}\partial_c\right)\right) \\
&= \frac{\partial y^{a'}}{\partial x^a}\mathrm{d}x^a\left(\left(\frac{\partial x^b}{\partial y^{b'}}\partial_b\left(\frac{\partial x^c}{\partial y^{c'}}\right)\right)\partial_c + \frac{\partial x^b}{\partial y^{b'}}\frac{\partial x^c}{\partial y^{c'}}(\nabla_{\partial_b}\partial_c)\right) \\
&= \frac{\partial y^{a'}}{\partial x^a}\frac{\partial^2 x^a}{\partial y^{b'}\partial y^{c'}} + \frac{\partial y^{a'}}{\partial x^a}\frac{\partial x^b}{\partial y^{b'}}\frac{\partial x^c}{\partial y^{c'}}\Gamma^a_{bc}
\end{aligned}$$

$$(\Gamma')^{a'}_{b'c'} = \frac{\partial y^{a'}}{\partial x^a}\frac{\partial x^b}{\partial y^{b'}} + \frac{\partial y^{a'}}{\partial x^a}\frac{\partial^2 x^a}{\partial y^{b'}\partial y^{c'}}$$

The first term transforms like a $(1, 2)$ tensor, but the second term does not, so the Christoffel symbols cannot be used to define a tensor field.

The anomalous term in the transformation of the Christoffel symbols will exactly cancel the anomalous transformation of the object $\partial_a X^b$!

Covariant derivatives

If we have an affine connection Γ, we can take *covariant derivatives* of vector fields as well as more general tensor fields.

The covariant derivative of a vector field X is a $(1,1)$ tensor field ∇X, defined by

$$\nabla X : (Y, \eta) \mapsto \eta(\nabla_Y X)$$

for all vector fields Y and covector fields X (check that this is C^∞-linear).
In abstract index notation, we write this tensor field as $\nabla_\mu X^\nu$. The components of the tensor field ∇X with respect to some local coordinates are written $\nabla_a X^b$. Note the positions of the indices.

The components of the covariant derivative

Expanding in terms of coordinate induced vector fields:

$$\begin{aligned}\nabla_Y X &= \nabla_{(Y^a\partial_a)}(X^b\partial_b)\\ &= Y^a\nabla_a(X^b\partial_b)\\ &= Y^a(\partial_a X^b)\partial_b + Y^a X^b \Gamma^c_{ab}\partial_c\\ &= Y^a\left(\partial_a X^b + \Gamma^b_{ac}X^c\right)\partial_b,\end{aligned}$$

where in the last line we have relabelled some of the dummy indices. Hence, the components of ∇X are

$$\nabla_a X^b = \partial_a X^b + \Gamma^b_{ac}X^c.$$

Don't take the position of these indices too seriously! Consider the component $\nabla_0 X^1$:

$$\begin{aligned}\nabla_0 X^1 &= \partial_0 X^1 + \Gamma^1_{0c} X^c \\ &= \partial_0 X^1 + \Gamma^1_{00} X^0 + \Gamma^1_{01} X^1 + \Gamma^1_{02} X^2 + \Gamma^1_{03} X^3.\end{aligned}$$

In general this is not an operator "∇_0" acting on the component X^1 - instead, it depends on *all* of the components of X. It should be thought of as the $(0,1)$ component of the $(1,1)$ tensor field ∇X.

Derivatives of general tensor fields

We can extend the covariant derivative to act on scalar fields, covector fields and higher rank tensor fields.

For a scalar field f, we define

$$\nabla f = \mathrm{d}f.$$

We now extend the covariant derivative to act on other tensors by requiring that it obeys the Leibniz rule for products.
So, for a covector field η and an arbitrary vector field X, in some arbitrary coordinate system we have

$$\begin{aligned}
\mathrm{d}_a\left(\eta_b X^b\right) &= \left(\nabla_a \eta_b\right) X^b + \eta_b\left(\nabla_a X^b\right) \\
\Rightarrow (\partial_a \eta_b) X^b + \eta_b(\partial_a X^b) &= \left(\nabla_a \eta_b\right) X^b + \eta_b(\partial_a X^b) + \eta_b \Gamma^b_{ac} X^c \\
\Rightarrow \left(\nabla_a \eta_b\right) X^b &= (\partial_a \eta_b - \Gamma^c_{ab}\eta_c) X^b.
\end{aligned}$$

Since this holds for *all* vector fields X, we must have

$$\nabla_a \eta_b = \partial_a \eta_b - \Gamma^c_{ab}\eta_c.$$

Following the same kind of reasoning, we can write out the formula for the covariant derivative of a tensor of general rank:

$$\begin{aligned}\nabla_a T^{b_1 \dots b_n}{}_{c_1 \dots c_m} = {} & \partial_a T^{b_1 \dots b_n}{}_{c_1 \dots c_m} \\ & + \Gamma^{b_1}_{ad} T^{d b_2 \dots b_n}{}_{c_1 \dots c_m} + \dots + \Gamma^{b_n}_{ad} T^{b_1 b_2 \dots d}{}_{c_1 \dots c_m} \\ & - \Gamma^{d}_{a c_1} T^{b_1 \dots b_n}{}_{d c_2 \dots c_m} - \dots - \Gamma^{d}_{a c_m} T^{b_1 \dots b_n}{}_{c_1 c_2 \dots d}.\end{aligned}$$