C7.5 Lecture 9: Differential geometry 5

The Levi-Civita connection, normal coordinates and parallel transport

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Last time we defined the Christoffel symbols Γ^a_{bc} associated with a given affine connection Γ (with associated covariant derivative ∇), but we did not actually construct an affine connection. We can easily see that many affine connections are possible:

- Choose some local coordinates, and pick some smooth functions Γ^a_{bc}. These will be the Christoffel symbols in the chosen coordinate system – e.g. for a vector field X, we define ∇X as the vector field with components

$$\nabla_a X^b = \partial_a X^b + \Gamma^b_{ac} X^c.$$

- The Christoffel symbols in any other coordinates can be found by using the transformation law for Christoffel symbols. From this transformation law, it follows that our definition of the tensor ∇X is independent of our original choice of coordinates.

The commutator

Which affine connection should we use? Before answering this, we need to define a tensor called the *torsion*, and before that we need to define the *commutator* of two vector fields X and Y.
The commutator $[X, Y]$ is the vector field which acts on scalar fields as

$$[X, Y](f) := X(Y(f)) - Y(X(f)).$$

This *does* define a vector field, since it satisfies linearity and the Leibniz rule. In components

$$[X, Y]^a = X^b \partial_b Y^a - Y^b \partial_b X^a.$$

We can construct the vector with these components in the chosen coordinate system, and then we can check that this definition is independent of the particular coordinate system. Note that the commutator can be defined without using an affine connection!

The torsion

The *torsion* of the connection ∇ is the $(1, 2)$ tensor field T, whose action on the covector field η and the vector fields X, Y is

$$T(\eta, X, Y) = \eta\left(\nabla_X Y - \nabla_Y X - [X, Y]\right).$$

In terms of components, this is (**exercise**)

$$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb}.$$

The Levi-Civita connection

The Levi-Civita connection is torsion free: $T^{\mu}_{\nu\rho} = 0$. In other words the Christoffel symbols are symmetric in their lower indices.
The other feature of the Levi-Civita connection is that it is *compatible with the metric*:

$$\nabla_{\mu} g_{\nu\rho} = 0$$

Using these two properties, we can calculate

$$\begin{aligned}
&\nabla_a g_{bc} + \nabla_b g_{ac} - \nabla_c g_{ab} \\
&= \partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab} \\
&\quad - \Gamma^d_{ab} g_{dc} - \Gamma^d_{ac} g_{bd} - \Gamma^d_{ba} g_{dc} - \Gamma^d_{bc} g_{ad} + \Gamma^d_{ca} g_{db} + \Gamma^d_{cb} g_{ad} \\
&= \partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab} - 2\Gamma^d_{ab} g_{dc}
\end{aligned}$$

$$\Rightarrow \Gamma^c_{ab} = \frac{1}{2}(g^{-1})^{cd} \left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\right)$$

The components of the Levi-Civita connection are given by the expression

$$\Gamma^{c}_{\ ab} = \frac{1}{2}(g^{-1})^{cd}\left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\right)$$

in *all* coordinate systems. You can check that this expression does *not* transform as a $(1,2)$ tensor field, but instead as the components of an affine connection.

From this point onwards, we will *always* work with the Levi-Civita connection.

Normal coordinates

We can *always* choose coordinates so that, at some point, the Christoffel symbols vanishes, i.e. given $p \in \mathcal{M}$, we can choose some local coordinates x^a in a neighbourhood of p such that

$$\Gamma^a_{bc}\Big|_p = 0.$$

These coordinates can also further chosen so that the components of the metric at p are

$$g_{ab}\Big|_p = \mathrm{diag}(-1, 1, 1, 1).$$

These coordinates are called *normal coordinates at p*.

Using normal coordinates can simplify a lot of computations - you should remember that, if some equation holds in a particular coordinate system, *and* if that equation can be written entirely in terms of tensors or tensor fields, then the equation must hold in all coordinate systems. E.g. if we derive an expression like

$$X^a = Y^b \nabla_b Z^a,$$

for the components relative to a system of normal coordinates, and if this equation also holds at an *arbitrary* point in the manifold, then we actually have an equality between vector fields:

$$X = \nabla_Y Z.$$

The proof that normal coordinates exist (given a torsion-free connection) are in the appendix to the lecture notes (non-examinable).

Parallel transport

A tensor field T is *parallel transported* (or "parallely transported") along the integral curves of the vector field X if

$$\nabla_X T = 0,$$

or, if you prefer abstract indices

$$X^\rho \nabla_\rho T^{\mu_1\mu_2\ldots\mu_n}{}_{\nu_1\nu_2\ldots\nu_m} = 0.$$

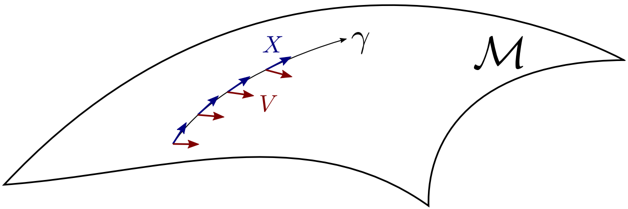
This is the closest we can get to saying that T remains "parallel to itself" when moved in the direction X.

Note, however, that the *components* of T in any particular coordinate system do not necessarily remain constant!

In normal coordinates at the point p the derivative of the components of T in the direction X vanishes:

$$X^c \partial_c T^{a_1 a_2 \dots a_n}{}_{b_1 b_2 \dots b_m} = 0$$

but this *only* holds at the point p. Note also that this is *not* a 'tensorial' equation, since it involves the "non-tensorial" operator ∂_c, so it is not true in a general coordinate system.



X is the tangent vector to the curve γ. Here, the vector field V is parallel transported along γ, i.e. $\nabla_X V = 0$.