C7.5 Lecture 10: Differential geometry 6

Geodesics

Joe Keir

Joseph.Keir@maths.ox.ac.uk

"Straight lines" played important physical roles in pre-GR theories:

- In Aristotelian/atomist spacetimes, "vertical" straight lines represented the paths of particles/observers "at rest".
- In Galilean spacetimes, any straight line that's transverse to the surfaces of "constant time" represents the path of a particle moving at a constant velocity, or an "inertial observer".
- In Minkowski spacetime, a straight line with tangent vector in the interior of the light cones represents represents the path of a particle moving at a constant velocity, or an "inertial observer".

In all cases, these special straight lines represent the paths of "test particles" on which no external forces are acting.

Generalising these ideas to curved spacetime is key to interpreting GR, but spacetime no longer has a vector space/affine space/vector bundle structure, so it is not obvious how to do this.

Fixing a vector V at a point p, and then "moving in that direction" (i.e. considering points $q(\lambda)$ s.t. $q - p = \lambda V$) is one way to generate straight lines in an affine space which *does not* easily generalise to a curved space.

Two ways which *do* generalise:

1. curves which "minimise distance", and
2. curves which "do not accelerate".

Let γ be a timelike curve through points p and q, parametrised so that $\gamma(0) = p$ and $\gamma(1) = q$. Work in local coordinates x^a, where $\gamma \circ \phi_U^{-1} = x^a(\lambda)$. Then the proper time along the curve is

$$\tau(p,q)[\gamma] := \int_0^1 \sqrt{-g_{ab}(x)\frac{\mathrm{d}x^a(\lambda)}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b(\lambda)}{\mathrm{d}\lambda}}\;\mathrm{d}\lambda.$$

To see this, change parameter from λ to τ, then we see that

$$\begin{aligned}\tau(p,q)[\gamma] &= \int_0^{\tau(p,q)[\gamma]} \sqrt{-g_{ab}(x)\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda}\right)^2}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\;\mathrm{d}\tau \\ &= \int_0^{\tau(p,q)[\gamma]} \mathrm{d}\tau.\end{aligned}$$

The Euler-Lagrange equations can be used to find the curves which extremise this integral. Defining the Lagrangian

$$\mathscr{L} := \sqrt{-g_{ab}(x)\frac{\mathrm{d}x^a(\lambda)}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b(\lambda)}{\mathrm{d}\lambda}}$$

Varying the path $x^a(\lambda)$, the Euler-Lagrange equations give (**exercise**)

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\lambda^2} + \frac{1}{2}(g^{-1})^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right)\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\frac{\mathrm{d}x^c}{\mathrm{d}\lambda} = \mathscr{L}^{-1}\frac{\mathrm{d}\mathscr{L}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$$

$$\Leftrightarrow \frac{\mathrm{d}^2x^a}{\mathrm{d}\lambda^2} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}\frac{\mathrm{d}x^c}{\mathrm{d}\lambda} = \mathscr{L}^{-1}\frac{\mathrm{d}\mathscr{L}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}.$$

The Levi-Civita connection arises naturally! In fact, using variational techniques (particularly looking for conserved quantities) applied to this Lagrangian is often the quickest and easiest way to derive the components of the Levi-Civita connection.

If we wish, we can choose the variable λ to be the proper time τ, in which case $\mathscr{L} \equiv 1$, and we obtain

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \Gamma^a_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}\tau} \frac{\mathrm{d}x^c}{\mathrm{d}\tau} = 0$$

Exactly the same equations can be derived for spacelike curves by extremising the proper length.

These equations are called the *geodesic equations* and solutions are called *geodesics*.

The geodesic equation is a second order ODE for the coordinates $x^a(\tau)$ of the geodesic. To solve it, we need both the initial position $x^a(0)$ and the initial tangent vector $\frac{\mathrm{d}x^a}{\mathrm{d}\tau}$.

We can also write this equation in terms of the tangent vector to the curve γ. The components of the tangent vector are $X^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}$, and so

$$\frac{\mathrm{d}X^a}{\mathrm{d}\tau} + \Gamma^a_{bc} X^b X^c = 0.$$

Along the curve γ (parametrised by proper time) with tangent vector X, for any function f we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} f = X^a \partial_a f = X(f),$$

so the geodesic equation can be written as

$$\begin{aligned} X^b \partial_b X^a + \Gamma^a_{bc} X^b X^c &= 0 \\ \Leftrightarrow X^b \nabla_b X^a &= 0 \end{aligned}$$

Sp we can write the geodesic equation in the 'tensorial' manner:

$$\nabla_X X = 0$$

in other words, X is parallel transported along its own integral curve.

This gives us a new way to think about geodesics: *a geodesic is a curve along which the tangent vector to the curve remains parallel to itself*. This is a bit like our first definition of a straight line in an affine space.
It also means that the curve does not accelerate. Define the acceleration vector along the curve

$$a = \nabla_X X.$$

In normal coordinates at a point p, this means that

$$a^a = \frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2},$$

but note that this is not a tensorial expression: in a general coordinate system,

$$a^a = \frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \Gamma^a_{bc}\left(\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}x^c}{\mathrm{d}\tau}\right).$$

This also allows us to define *null geodesics*: a null geodesic is a curve whose tangent vector X is both null and satisfies $\nabla_X X = 0$.

Any curve satisfying $\nabla_X X = 0$ is said to be *affinely parametrised* (it is non-affinely parametrised, but still a geodesic, if it satisfies $\nabla_X X = fX$ for some scalar function f).

If a geodesic is initially timelike/spacelike/null, then it will remain so: let λ be an affine parameter, then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(X^\mu X_\mu\right) &= \nabla_X \left(X^\mu X_\mu\right) \\ &= 2 g_{\mu\nu} X^\nu (\nabla_X X)^\mu \\ &= 0 \end{aligned}$$

where we have used the fact that $\nabla g = 0$. So $g(X, X)$ is constant along a geodesic.

Finally, note that an alternative Lagrangian, which only works for affinely parametrised null geodesics, is

$$\mathscr{L} = g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda}.$$

We can check that, varying this, we obtain the same Euler-Lagrange equations as we obtained originally (with $\mathscr{L} =$ constant, which is another way to define an affine parameter).

We can choose to parametrise by proper time in the timelike case, in which $\mathscr{L} = -1$, or proper length in the spacelike case, in which case $\mathscr{L} = 1$. There are no privileged affine parameters in the null case.

This formulation has two advantages: it is easier to work with (no square roots!), and it works for null geodesics (we don't divide by $\mathscr{L}$).

Interpretation

Just as in pre-GR theories, geodesics are interpreted as the paths of test particles or observers who do not experience any external forces. If the particles is massive then it will follow a timelike geodesic, if it is massless then it will follow a null geodesic.

It is important that these test particles are "small" – large bodies can experience tidal forces. As we will see later, energy-momentum will itself *cause* curvature of spacetime, and thus influence the behaviour of geodesics, so it is also important that the test particle has negligible energy-momentum. In this case we might say that the particle moves along geodesics in the "background spacetime", or that we are "neglecting back-reaction".