C7.5 Special Lecture 10.5 (examples 2)

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The torsion is a tensor

Recall from lectures that the "torsion tensor" is the map

$$T(\eta, X, Y) = \eta(\nabla_X Y - \nabla_Y X - [X, Y]),$$

where η is a covector field and X, Y are vector fields.
To prove that this is a tensor, we need to show C^∞ linearity in all arguments. Linearity in the first argument (η) is obvious. Also, the expression is obviously antisymmetric in its second and third arguments, so it is sufficient to prove linearity in the second argument.

For a covector field η, scalar field a and vector fields X, X' and Y we can calculate

$$T(\eta, aX + X', Y) = \eta\left(\nabla_{aX+X'}Y - \nabla_Y(aX + X') - [aX + X', Y]\right)$$

$$= \eta\big(a\nabla_X Y + \nabla_{X'}Y - a\nabla_Y X - Y(a)X - \nabla_Y X' - a[X, Y] + Y(a)X - [X', Y]\big)$$

$$= a\eta(\nabla_X Y - \nabla_Y X - [X, Y]) + \eta(\nabla_{X'}Y - \nabla_Y X' - [X', Y'])$$

$$= aT(\eta, X, Y) + T(\eta, X', Y).$$

Curves and their properties on a model spacetime

Consider the model three-dimensional spacetime $\mathbb{R} \times \mathbb{S}^2$, with metric

$$g = -\mathrm{d}t^2 + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2,$$

where $t \in \mathbb{R}$, $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$. Note that these coordinates don't cover the whole sphere – we would need to use another set of coordinates to cover the line $\phi = 0$ and the two poles. But in this question we'll just work in the region covered by these coordinates.

Consider a curve γ, given in local coordinates (t, θ, ϕ) (i.e. we give $\tilde{\gamma} = \gamma \circ \phi_U^{-1}$) as

$$\tilde{\gamma}(\lambda) = (\lambda, a(\lambda), b(\lambda)) \,.$$

The tangent vector to this curve is

$$V = \partial_t + \frac{\mathrm{d}a}{\mathrm{d}\lambda}\partial_\theta + \frac{\mathrm{d}b}{\mathrm{d}\lambda}\partial_\phi = \partial_t + \dot{a}\partial_\theta + \dot{b}\partial_\phi,$$

so we can calculate

$$g(V, V) = -1 + \dot{a}^2 + \sin^2\theta\,\dot{b}^2.$$

Now suppose that the curve γ is timelike, so

$$-1 + \dot{a}^2 + (\sin^2 a)\dot{b}^2 < 0.$$

Then we can change from the parameter λ to proper time. Consider a change of parametrisation $\lambda = \lambda(\tau)$; then we can see that the new tangent vector is

$$V' = \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\left(\partial_t + \dot{a}\partial_\theta + \dot{b}\partial_\phi\right),$$

so, to change to proper time, we need to solve the equation

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2 = \frac{1}{1 - \dot{a}^2 - (\sin^2 a)\dot{b}^2},$$

which we can also write as

$$\frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \sqrt{1 - \dot{a}^2 - (\sin^2 a)\dot{b}^2},$$

where we recall that a and b are functions of λ.

For example: suppose that $a(\lambda) \equiv \frac{\pi}{2}$ and $b(\lambda) = v\lambda$ for some $v < 1$. Then the curve, parametrised by proper time, is given in local coordinates by

$$\tilde{\gamma}(\tau) = \left(\frac{\tau}{\sqrt{1 - v^2}}, \frac{\pi}{2}, \frac{v\tau}{\sqrt{1 - v^2}} \right).$$

Geodesics and Christoffel symbols

Next, let's work out the geodesic equation (and solve it), and simultaneously derive the Christoffel symbols relative to our chosen local coordinates.
The easiest way to do this is to work with the Lagrangian

$$\mathscr{L} = g_{ab}\dot{x}^a\dot{x}^b,$$

where the dots are derivatives w.r.t. an affine parameter. Hence

$$\mathscr{L} = -\dot{t}^2 + \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2.$$

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We can immediately find some conserved quantities:

- $\mathscr{L}$ is independent of t, so $E := -\frac{1}{2}\frac{\partial\mathscr{L}}{\partial\dot{t}} = \dot{t}$ is constant along geodesics.
- $\mathscr{L}$ is independent of ϕ, so $\Omega = \frac{1}{2}\frac{\partial\mathscr{L}}{\partial\dot{\phi}} = \sin^2\theta\dot{\phi}$ is constant along geodesics.
- $\mathscr{L}$ itself is constant: it can be set to either -1, 0 or 1 in the timelike/null/spacelike cases.

Let's concentrate on timelike geodesics. Substituting the conserved quantities in to the equation for $\mathscr{L}$, we find

$$-1 = -E^2 + \dot{\theta}^2 + \frac{\Omega^2}{\sin^2\theta},$$

where now the dots are derivatives w.r.t. proper time τ. You can solve this for $\theta(\tau)$, but it's a bit of a mess. You can understand the motion by considering a particle moving in a 1-dimensional potential, with unit mass, potential $V(\theta) = \frac{\Omega^2}{2\sin^2\theta}$ and energy $\frac{1}{2}(E^2 - 1)$:

$$\frac{1}{2}\dot{\theta}^2 + \frac{\Omega^2}{2\sin^2\theta} = \frac{1}{2}(E^2 - 1).$$

To work out the Christoffel symbols, remember that the Euler-Lagrange equations will be of the form

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0.$$

First consider the E-L equation for t: this is

$$\ddot{t} = 0,$$

so we conclude that $\Gamma^t_{ab} = 0$. Next consider the E-L equation for θ: this is

$$\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0,$$

so we have $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$, and $\Gamma^\theta_{ab} = 0$ otherwise. Finally consider the E-L equation for ϕ:

$$\sin^2\theta\ddot{\phi} + 2(\sin\theta\cos\theta)\dot{\phi}\dot{\theta} = 0,$$

and so we have $\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta$, and $\Gamma^\phi_{ab} = 0$ otherwise.