C7.5 Lecture 11: Differential geometry 7

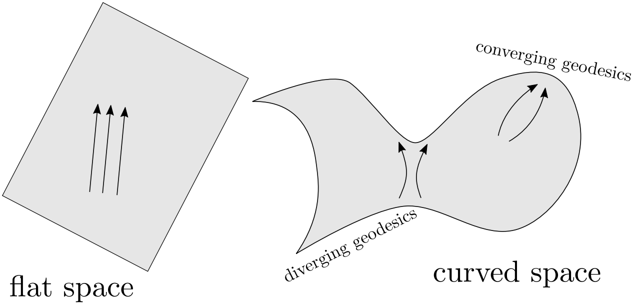
Curvature

Joe Keir

Joseph.Keir@maths.ox.ac.uk

Everything we've done so far has allowed us to reconstruct the tools we need to do physics on a curved spacetime. Finally, we are in a position to investigate the new aspect of curved spacetimes – curvature! Eventually we will link curvature to gravity, via the Einstein equations.

Curvature can be thought of as a measure of the deviation of a manifold from flat space. There are many ways to get a handle on this: we will use *geodesic deviation*. The idea is that curvature can cause nearby geodesics to converge or diverge.

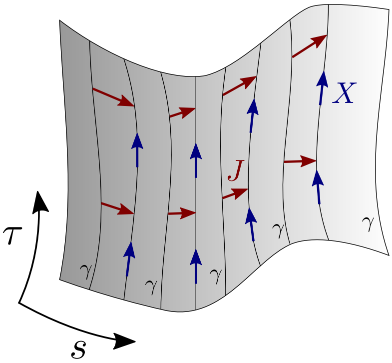


Curvature can cause nearby “parallel” geodesics to either converge or diverge.

Consider a one-parameter family of timelike geodesics, given in local coordinates by $x^a(\tau) = \gamma^a(\tau, s)$. Here τ is the proper time along each geodesic, while the geodesics are labelled by a continuous parameter s, so different values of s correspond to different geodesics.
The tangent vector to the curve $\gamma(\cdot, s)$ is X, with components X^a. We can also define a vector J. This is not a vector field, but a vector defined along each of the curves γ, with components

$$J^a = \left.\frac{\partial \gamma^a}{\partial s}\right|_\tau$$

J is sometimes called a *deviation vector* or a *Jacobi field*.



A congruence of timelike geodesics $\gamma(\tau, s)$, where τ is the proper time along a geodesic and s labels the different geodesics. Note that this congruence (locally) defines a 2 dimensional surface in spacetime. At each point on this surface, X is the tangent vector to the geodesic and J is a Jacobi field, which commutes with X. The acceleration of J along a timelike geodesic measures the *geodesic deviation*.

Along each geodesic we can choose the origin of the proper time, i.e. the point at which $\tau = 0$. Under the (s-dependent) reparametrisation $\tau(s) \mapsto \tau(s) + b(s)$, we have

$$X \mapsto X$$
$$J \mapsto J + b'X$$

We can use this to ensure that, at $\tau = 0$, we have $g(J, X) = 0$ (note that $g(X, X) = -1$).

The vectors X and J commute:

$$\begin{aligned}[X, J]^a &= X^b \partial_b J^a - J^b \partial_b X^a \\ &= \frac{\partial}{\partial \tau} J^a - \frac{\partial}{\partial s} X^a \\ &= \frac{\partial^2 x^a}{\partial \tau \partial s} - \frac{\partial^2 x^a}{\partial s \partial \tau} \\ &= 0.\end{aligned}$$

The value of $g(X, J)$ is also constant along the geodesics:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau} g(X, J) &= \nabla_X \left(g_{\mu\nu} X^\mu J^\nu\right) \\ &= X_\mu \nabla_X J^\mu \\ &= X_\mu \nabla_J X^\mu + X_\mu [X, J]^\mu \\ &= \frac{1}{2} \nabla_J (X_\mu X^\mu) \\ &= \frac{1}{2} \nabla_J (-1) = 0\end{aligned}$$

so, since $g(X, J) = 0$ at $\tau = 0$ along each geodesic, X and J will remain orthogonal everywhere they are defined.

We can think of J as a "connecting vector", measuring the infinitesimal displacement of the geodesic $\gamma(\cdot, s + \epsilon)$ from the geodesic $\gamma(\cdot, s)$.

Next we compute the “acceleration” of the Jacobi field J along each of the geodesics. We can think of this as measuring the acceleration of an infinitesimally displaced geodesic.

$$\frac{\partial^2}{\partial \tau^2} J = \nabla_X \nabla_X J$$

$$= \nabla_X \nabla_J X \qquad \text{(using the torsion-free property and the fact that } J \text{ and } X \text{ commute)}$$

$$= \left(\nabla_X \nabla_J - \nabla_J \nabla_X - \nabla_{[X,J]} \right) X$$

(using the geodesic equation $\nabla_X X = 0$ and that J and X commute)

Why did we add additional terms which vanish? The point is that the object in the final line can be used to define a tensor field. First define the vector field

$$R(X, Y)Z := \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z$$

where X, Y and Z are vector fields. For fixed X, Y, $R(X, Y)$ can be thought of as a map (depending on the vector fields X and Y) from vector fields to vector fields, taking Z to $R(X, Y)Z$.

We'll show that this map is C^∞-linear in each of its arguments.

First, note that it is antisymmetric in its first two arguments:

$$R(X,Y)Z = -R(Y,X)Z$$

so we only need to check linearity in one of these arguments:

$$\begin{aligned}
&R(aX+bX',Y)Z \\
&= \left(a\nabla_X\nabla_Y + b\nabla_{X'}\nabla_Y - \nabla_Y(a\nabla_X + b\nabla_{X'}) - \nabla_{[aX+bX',Y]}\right) Z \\
&= \Big(a\nabla_X\nabla_Y + b\nabla_{X'}\nabla_Y - a\nabla_Y\nabla_X - b\nabla_Y\nabla_{X'} - (Y(a))\nabla_X - (Y(b))\nabla_{X'} \\
&\qquad - \nabla_{a[X,Y]-Y(a)X+b[X',Y]-Y(b)X'}\Big) Z \\
&= \left(a\nabla_X\nabla_Y - a\nabla_Y\nabla_X - a\nabla_{[X,Y]} + b\nabla_{X'}\nabla_Y - b\nabla_Y\nabla_{X'} - b\nabla_{[X',Y]}\right) Z \\
&= aR(X,Y)Z + bR(X',Y)Z
\end{aligned}$$

so $R(X,Y)Z$ is C^∞ linear in X and Y.

$R(X, Y)$ is also a C^∞ linear map from vector fields to vector fields. Easy to see that $R(X, Y)(Z + Z') = R(X, Y)Z + R(X, Y)Z'$, so we only need to check

$$\begin{aligned} R(X, Y)(aZ) &= (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(aZ) \\ &= a\nabla_X \nabla_Y Z + X(a)\nabla_Y Z + Y(a)\nabla_X Z + X(Y(a))Z \\ &\quad - a\nabla_Y \nabla_X Z - Y(a)\nabla_X Z - X(a)\nabla_Y Z - Y(X(a))Z \\ &\quad - a\nabla_{[X,Y]} Z - ([X, Y](a))\, Z \\ &= a\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right) \\ &\quad + (X(Y(a)) - Y(X(a)) - [X, Y](a)) \\ &= aR(X, Y)Z \end{aligned}$$

so $R(X, Y)$ is a linear map from vector fields to vector fields, which is also linear in both X and Y.

We can use this to define a $(1,3)$ tensor field, the *Riemann curvature tensor* R (or just the *Riemann tensor*). Acting on a covector field η and three vector fields X, Y, Z,

$$R(\eta, Z, X, Y) := \eta\,(R(X,Y)Z)$$

(If you know some differential geometry, this is the generalisation of the Gauss curvature from 2 dimensional surfaces.)

Returning now to the equation governing the acceleration of Jacobi fields, we find that a Jacobi field satisfies the ODE

$$\frac{\partial^2}{\partial \tau^2} J = R(X, J)X$$

so it is the Riemann curvature which governs the deviation of nearby geodesics. In flat space, the Riemann curvature vanishes!

You might prefer to work in abstract indices. Working through everything in components, we find that, for any vector field X (**exercise**)

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) X^\alpha = R^\alpha{}_{\beta\mu\nu} X^\beta.$$

We have seen that, when applied to a vector field, the commutator $[\nabla_\mu, \nabla_\nu]$ gives us back the Riemann curvature tensor. What happens when we apply this to other types of tensor field?

If f is a scalar field, then working in local coordinates

$$\begin{aligned}
[\nabla_a, \nabla_b]f &= \nabla_a \partial_b f - \nabla_b \partial_a f \\
&= \partial_a \partial_b f - \partial_b \partial_a f + \Gamma^c_{ab} \partial_c f - \Gamma^c_{ba} \partial_c f \\
&= 0
\end{aligned}$$

using the torsion-free property of the connection.

Together with the Leibniz rule, this allows us to work out the action of $[\nabla_\mu, \nabla_\nu]$ on a covector field

$$[\nabla_\mu, \nabla_\nu]\eta_\rho = -R^\sigma{}_{\rho\mu\nu}\eta_\sigma,$$

and then for a general tensor

$$\begin{aligned}
&[\nabla_\mu, \nabla_\nu] T^{\rho_1\rho_2\ldots\rho_n}{}_{\sigma_1\sigma_2\ldots\sigma_m} \\
&= R^{\rho_1}{}_{\kappa\mu\nu} T^{\kappa\rho_2\ldots\rho_n}{}_{\sigma_1\sigma_2\ldots\sigma_m} + R^{\rho_2}{}_{\kappa\mu\nu} T^{\rho_1\kappa\ldots\rho_n}{}_{\sigma_1\sigma_2\ldots\sigma_m} \\
&\quad + \ldots + R^{\rho_n}{}_{\kappa\mu\nu} T^{\rho_1\rho_2\ldots\kappa}{}_{\sigma_1\sigma_2\ldots\sigma_m} \\
&\quad - R^{\kappa}{}_{\sigma_1\mu\nu} T^{\rho_1\rho_2\ldots\rho_n}{}_{\kappa\sigma_2\ldots\sigma_m} - R^{\kappa}{}_{\sigma_2\mu\nu} T^{\rho_1\rho_2\ldots\rho_n}{}_{\sigma_1\kappa\ldots\sigma_m} \\
&\quad - \ldots - R^{\kappa}{}_{\sigma_m\mu\nu} T^{\rho_1\rho_2\ldots\rho_n}{}_{\sigma_1\sigma_2\ldots\kappa}.
\end{aligned}$$