C7.5 Lecture 12: Differential geometry 8

More about the curvature tensor

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Algebraic symmetries of the Riemann tensor

$$R(\eta, Z, X, Y) = \eta\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right)$$

It is easy to see that $R(\eta, Z, X, Y) = -R(\eta, Z, Y, X)$. From this it follows that the Riemann tensor is antisymmetric in its last two indices:

$$R^{\mu}{}_{\nu\rho\sigma} = -R^{\mu}{}_{\nu\sigma\rho}.$$

From the metric compatibility condition we also find

$$\begin{aligned} 0 &= -[\nabla_\mu, \nabla_\nu] g_{\rho\sigma} \\ &= R^\alpha{}_{\rho\mu\nu} g_{\alpha\sigma} + R^\alpha{}_{\sigma\mu\nu} g_{\rho\alpha} \\ &= R_{\sigma\rho\mu\nu} + R_{\rho\sigma\mu\nu}, \end{aligned}$$

so the Riemann tensor is also antisymmetric in its first two indices.

Next consider the following expression, for some scalar field f

$$\nabla_\mu \nabla_\sigma \nabla_\nu f + \nabla_\sigma \nabla_\nu \nabla_\mu f + \nabla_\nu \nabla_\mu \nabla_\sigma f$$
$$- \nabla_\sigma \nabla_\mu \nabla_\nu f - \nabla_\mu \nabla_\nu \nabla_\sigma f - \nabla_\nu \nabla_\sigma \nabla_\mu f.$$

We can group these terms in two different ways: first,

$$\nabla_\mu \left([\nabla_\sigma, \nabla_\nu] f \right) + \nabla_\nu \left([\nabla_\mu, \nabla_\sigma] f \right) + \nabla_\sigma \left([\nabla_\nu, \nabla_\mu] f \right) = 0.$$

On the other hand,

$$[\nabla_\sigma, \nabla_\nu] \nabla_\mu f + [\nabla_\mu, \nabla_\sigma] \nabla_\nu f + [\nabla_\nu, \nabla_\mu] \nabla_\sigma f$$
$$= \left(R^\alpha{}_{\mu\nu\sigma} + R^\alpha{}_{\nu\sigma\mu} + R^\alpha{}_{\sigma\mu\nu} \right) \nabla_\alpha f.$$

But since this holds for *all* scalars f, we have the *first Bianchi identity* or the *algebraic Bianchi identity*:

$$R^\alpha{}_{\mu\nu\sigma} + R^\alpha{}_{\nu\sigma\mu} + R^\alpha{}_{\sigma\mu\nu} = 0.$$

Another useful symmetry of the Riemann tensor follows from the symmetries we already know. Using the first Bianchi identity and cyclicly permuting indices, we have

$$
\begin{aligned}
R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} &= 0 \\
-R_{\nu\rho\sigma\mu} - R_{\nu\sigma\mu\rho} - R_{\nu\mu\rho\sigma} &= 0 \\
-R_{\rho\sigma\mu\nu} - R_{\rho\mu\nu\sigma} - R_{\rho\nu\sigma\mu} &= 0 \\
R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} + R_{\sigma\rho\mu\nu} &= 0.
\end{aligned}
$$

Adding these and using antisymmetry in the first and last pair of indices, we obtain

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}.$$

Summary of algebraic symmetries of the Riemann tensor

$$
\begin{aligned}
R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho} \\
R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma} \\
R_{\mu\nu\rho\sigma} &= R_{\rho\sigma\mu\nu} \\
R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} &= 0.
\end{aligned}
$$

The (second) Bianchi identity

There is also an important symmetry of the *derivatives* of the Riemann tensor, called the *second Bianchi identity* or simply the *Bianchi identity*.
Consider the following expression, for some arbitrary covector η

$$\begin{aligned}&\nabla_\mu\nabla_\rho\nabla_\nu\eta_\sigma + \nabla_\rho\nabla_\nu\nabla_\mu\eta_\sigma + \nabla_\nu\nabla_\mu\nabla_\rho\eta_\sigma \\ &- \nabla_\mu\nabla_\nu\nabla_\rho\eta_\sigma - \nabla_\nu\nabla_\rho\nabla_\mu\eta_\sigma - \nabla_\rho\nabla_\mu\nabla_\nu\eta_\sigma .\end{aligned}$$

Grouping the terms in one way we obtain

$$\begin{aligned}&[\nabla_\mu,\nabla_\rho]\nabla_\nu\eta_\sigma + [\nabla_\rho,\nabla_\nu]\nabla_\mu\eta_\sigma + [\nabla_\nu,\nabla_\mu]\nabla_\rho\eta_\sigma \\ &= R^\alpha{}_{\nu\rho\mu}\nabla_\alpha\eta_\sigma + R^\alpha{}_{\sigma\rho\mu}\nabla_\nu\eta_\alpha + R^\alpha{}_{\mu\nu\rho}\nabla_\alpha\eta_\sigma \\ &\quad + R^\alpha{}_{\sigma\nu\rho}\nabla_\mu\eta_\alpha + R^\alpha{}_{\rho\mu\nu}\nabla_\alpha\eta_\sigma + R^\alpha{}_{\sigma\mu\nu}\nabla_\rho\eta_\alpha \\ \\ &= \left(R^\alpha{}_{\mu\nu\rho} + R^\alpha{}_{\nu\rho\mu} + R^\alpha{}_{\rho\mu\nu}\right)\nabla_\alpha\eta_\sigma \\ &\quad + R^\alpha{}_{\sigma\nu\rho}\nabla_\mu\eta_\alpha + R^\alpha{}_{\sigma\rho\mu}\nabla_\nu\eta_\alpha + R^\alpha{}_{\sigma\mu\nu}\nabla_\rho\eta_\alpha \\ \\ &= R^\alpha{}_{\sigma\nu\rho}\nabla_\mu\eta_\alpha + R^\alpha{}_{\sigma\rho\mu}\nabla_\nu\eta_\alpha + R^\alpha{}_{\sigma\mu\nu}\nabla_\rho\eta_\alpha ,\end{aligned}$$

where we have used the first Bianchi identity.

But, grouping the same terms in an alternative way, we have

$$\begin{aligned}
&\nabla_\mu[\nabla_\rho, \nabla_\nu]\eta_\sigma + \nabla_\nu[\nabla_\mu, \nabla_\rho]\eta_\sigma + \nabla_\rho[\nabla_\nu, \nabla_\mu]\eta_\sigma \\
&= \nabla_\mu \left(R^\alpha{}_{\sigma\nu\rho}\eta_\alpha\right) + \nabla_\nu \left(R^\alpha{}_{\sigma\rho\mu}\eta_\alpha\right) + \nabla_\rho \left(R^\alpha{}_{\sigma\mu\nu}\eta_\alpha\right) \\
&= R^\alpha{}_{\sigma\nu\rho}\nabla_\mu\eta_\alpha + R^\alpha{}_{\sigma\rho\mu}\nabla_\nu\eta_\alpha + R^\alpha{}_{\sigma\mu\nu}\nabla_\rho\eta_\alpha \\
&\quad + \nabla_\mu R^\alpha{}_{\sigma\nu\rho}\eta_\alpha + \nabla_\nu R^\alpha{}_{\sigma\rho\mu}\eta_\alpha + \nabla_\rho R^\alpha{}_{\sigma\mu\nu}\eta_\alpha.
\end{aligned}$$

Combining these two equations and reordering the indices a bit using the symmetries of the Riemann tensor, we find that

$$\left(\nabla_\mu R_{\nu\rho}{}^\alpha{}_\sigma + \nabla_\nu R_{\rho\mu}{}^\alpha{}_\sigma + \nabla_\rho R_{\mu\nu}{}^\alpha{}_\sigma\right)\eta_\alpha = 0.$$

Since this holds for *all* covectors η (and since the connection is metric-compatible), we have the *second Bianchi identity*

$$\nabla_\mu R_{\nu\rho\alpha\beta} + \nabla_\nu R_{\rho\mu\alpha\beta} + \nabla_\rho R_{\mu\nu\alpha\beta} = 0.$$

Contractions of the Riemann tensor

Contracting a pair of indices in the Riemann tensor forms the *Ricci curvature tensor* (or simply *Ricci tensor*), which is also conventionally notated with the letter R:

$$R_{\mu\nu} := R^{\alpha}{}_{\mu\alpha\nu}$$

The symmetries of the Riemann tensor imply that the Ricci tensor is symmetric (**exercise**).

We can contract the indices of the Ricci tensor to form the *scalar curvature* or *Ricci scalar*, also conventionally denoted[1] by the letter R:

$$R := (g^{-1})^{\mu\nu} R_{\mu\nu}$$

[1] Because of these conventional notations, when dealing with the curvature it is particularly useful to use abstract index notation rather than index-free notation!

The Einstein tensor

If we contract indices in the second Bianchi identity, we obtain the identity

$$\nabla^\alpha R_{\alpha\mu\nu\rho} - \nabla_\nu R_{\mu\rho} + \nabla_\rho R_{\mu\nu} = 0.$$

Contracting again, this time with the indices μ and ρ (and relabelling indices and dividing by two), we obtain the *contracted Bianchi identity*

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = 0.$$

This leads us to define the *Einstein tensor*:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu},$$

which is *divergence free*

$$\nabla^\mu G_{\mu\nu} = 0.$$

Curvature in terms of the metric

One final aspect of the curvature which is important is its relationship to the metric tensor. In local coordinates x^a,

$$
\begin{aligned}
[\nabla_a, \nabla_b]X^c &= R^c{}_{dab}X^d \\
&= \partial_a \nabla_b X^c - \Gamma^d_{ab}\nabla_d X^c + \Gamma^c_{ad}\nabla_b X^d \quad - (a \leftrightarrow b) \\
&= \partial_a\partial_b X^c + \partial_a\left(\Gamma^c_{bd}X^d\right) - \Gamma^d_{ab}\partial_d X^c + \Gamma^c_{ad}\partial_b X^d \\
&\quad - \Gamma^d_{ab}\Gamma^c_{de}X^e + \Gamma^c_{ad}\Gamma^d_{be}X^e \quad - (a \leftrightarrow b) \\
&= \left(\partial_a\Gamma^c_{bd} - \partial_b\Gamma^c_{ad} + \Gamma^c_{ae}\Gamma^e_{bd} - \Gamma^c_{be}\Gamma^e_{ad}\right)X^d.
\end{aligned}
$$

Hence the components of the Riemann tensor can be written in terms of the Christoffel symbols and their derivatives:

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{bc}.$$

Recalling the expression for the Christoffel symbols of the Levi-Civita connection in terms of the metric components

$$\Gamma^a_{bc} = \frac{1}{2}(g^{-1})^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right),$$

Substituting into the previous equation, we obtain a long and not very enlightening equation for the components of the Riemann tensor.

The important thing to notice about this expression is the following: **the Riemann tensor depends on the metric g and its first two derivatives**. In normal coordinates, only the second derivatives of the metric survive.