# C7.5 Lecture 13: The Einstein equations

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# The Einstein equations

Remember that there are many hints that matter causes spacetime to curve, e.g. from the equivalence principle we saw that gravity isn't a force, and freely falling observers/test particles/particles experiencing no forces should move along geodesics, but that in this case geodesics cannot be ordinary straight lines.

To be consistent with special relativity, it should not be just the matter density which affects curvature, but the *energy* density. This appears in the energy-momentum tensor $T_{\mu\nu}$.

$T_{\mu\nu}$ is a symmetric tensor, and it somehow has to be related to the curvature. With this in mind, in October 1915 Einstein tried the equation

$$R_{\mu\nu} = CT_{\mu\nu}$$

for some constant $C$.

But there is a problem with this equation: the conservation of energy-momentum means that the energy momentum tensor $T_{\mu\nu}$ is divergence-free[1], while the divergence of the Ricci tensor generally does not vanish in general.

[1] To generalise all of our arguments from Minkowski space, we would need to understand integration on manifolds, which is beyond the scope of this course. The upshot is that everything has the same interpretation, if we replace partial derivatives with covariant derivatives.

By November 1915, Einstein had remedied this problem in the obvious way:

$$G_{\mu\nu} = CT_{\mu\nu}.$$

We still need to fix the constant $C$. This can be done by taking the *weak field limit*, where we take the metric to have the form

$$g_{ab} = m_{ab} + \epsilon h_{ab}$$

in some coordinates. We then expand the Einstein equations up to first order in $\epsilon$ and then compare the Einstein equations with Newtonian gravity (see the *GR2 course* for the details). The upshot of this calculation is that $C = 8\pi$. This leads to the *Einstein equations* (or the *Einstein field equations*)

$$G_{\mu\nu} = 8\pi T_{\mu\nu}.$$

Restoring the speed of light and Newton's constant, this is

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

# Uniqueness of the Einstein equations

There is a sense in which the Einstein equations are "unique". This is given by *Lovelock's theorem* (the proof of which is well beyond the scope of this course)

**Lovelock's theorem**

*In four spacetime dimensions, the only tensor fields constructed entirely from the metric tensor together with its first and second derivatives which are symmetric and divergence-free are of the form*

$$aG_{\mu\nu} + bg_{\mu\nu}$$

*where a and b are constants.*

This suggests the following alternative for the Einstein equations, which Einstein published in 1917

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

The constant $\Lambda$ in this equation is called the *cosmological constant*. These equations are sometimes called the *Einstein equations with cosmological constant*.

Einstein originally included the cosmological constant because, without it, he couldn't find cosmological solutions of the Einstein equations which didn't either expand or contract. When later observations showed that the universe *is* expanding, Einstein called it his "greatest mistake". More recent observations indicate that the cosmological constant is nonzero, but with an incredibly small positive value: in Planck units, $\Lambda \approx 7.26 \times 10^{-121}$.

# The Einstein equations as a system of PDEs

In a local system of coordinates, the Riemann curvature tensor can be written in terms of the metric and its first two derivatives. Hence the Einstein equations can be viewed as a second order system of PDEs for the metric components $g_{ab}$.

Usually we have to supplement these equations with the equations of motion for the matter in order to obtain a closed system of equations. But there is a special case, called the *Einstein vacuum equations*, where there is no matter present:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0.$$

In Newtonian gravity, when there is no matter present, the gravitational potential $\phi$ vanishes and there are no dynamics. The situation is very different in GR: the Einstein vacuum equations have many nontrivial solutions!

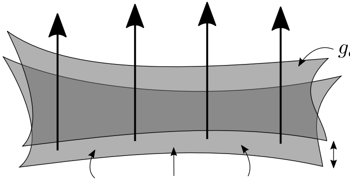
How do we solve these equations, in general? As with other second order PDEs, it depends on the character: are they

- *elliptic* like the Laplace equation,
- *parabolic* like the heat equation,
- *hyperbolic* like the wave equation?

Unfortunately, the Einstein equations are not of any specific type. What's more, if we try to view these equations as evolution equations, e.g. specifying the metric components $g_{ab}$ and their time derivatives everywhere on some initial time surface, then we find that *there is no unique solution*. Disaster!

What went wrong? Remember that we are viewing the equations as a system of PDEs for the components of the metric $g_{ab}$ in some coordinate system $x^a$. Suppose that $g_{ab}$ is a solution to the Einstein equations. But now consider another coordinate system $y^{a'}$, *which agrees with the coordinate system* $x^a$ *in a neighbourhood of the initial hypersurface*.

The metric in these coordinates has components $g'_{a'b'}$, which agree with the components $g_{ab}$ in a neighbourhood of the initial surface, but which will generally *differ* away from this neighbourhood. So there is no chance of a unique solution!




Initial data for the metric is given on the bottom surface, and we try to solve 'upwards'. One solution to the Einstein equations, which agrees with this initial data, is given by the metric $g_{ab}$. Another metric is given by $g'_{a'b'}$, whose components agree with the components of $g_{ab}$ on the initial hypersurface and for some amount of 'time', but then disagree at later times. However, they only differ by a coordinate transformation, so $g'_{a'b'} = \frac{\partial x^a}{\partial y^{a'}} \frac{\partial x^b}{\partial x^{b'}} g_{ab}$, where $x^a = y^a$ near the initial hypersurface. Since the Einstein equations are invariant under coordinate transformations, both $g_{ab}$ and $g_{a'b'}$ are solutions to the Einstein equations – so the solutions cannot be unique!

If $g_{ab}$ solves the Einstein equations, then so does $g'_{a'b'}$, so we can't have uniqueness. But the solution is obvious: we must choose a way of uniquely specifying coordinates.

One convenient choice is to choose the coordinate functions to satisfy wave equations ("wave coordinates"), in which case the Einstein equations become a *system of nonlinear wave equations*. Now we see that the Einstein equations are hyperbolic, and we can solve the *Cauchy problem*: pose initial data on some initial hypersurface, and then use the Einstein equations to evolve this data into the future.

Amazingly, it wasn't until almost 40 years after the Einstein equations were published when these issues were fully understood and solved (by Choquet-Bruhat in 1952). Although the details of these calculations are not necessary for our purposes, this point of view – viewing the Einstein equations as a system of evolution equations – is crucial both for a rigorous approach to GR, and for numerical GR.

# Other physical laws in curved spacetimes

## Point particles

The kinematics of point particles is governed by the **geodesic postulate**: *massive "test particles" move along timelike geodesics in the absence of external forces*. In general, however, we should avoid working with point masses in general relativity:

- taking some matter with a fixed mass and squeezing it into a smaller and smaller volume, GR predicts that at some point it will form a black hole, not a point mass!
- The energy-momentum of the particle should be included on the right hand side of the Einstein equations ("back reaction"), and so it will affect the metric and hence geodesics.

For these reasons, "test particles" in GR really just *mean* things following timelike geodesics. Sometimes we will talk about massless test particles – these just follow null geodesics.

# Other physical laws

At any point $p$ on the manifold, we can choose to work in normal coordinates at $p$. In this case, at $p$,

- the metric components are the same as the Minkowski metric components, and
- the Christoffel symbols vanish, so partial derivatives and (first) covariant derivatives are the same.

This means that, if we work in normal coordinates, then close to the point $p$ physics should look like physics in Minkowski space.

As an example, consider Maxwell's equations, which can be written (in inertial coordinates) in special relativity as

$$\begin{aligned} \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} &= 0 \\ (m^{-1})^{ab} \partial_a F_{bc} &= 0. \end{aligned}$$

These equations should take an identical form form in normal coordinates at the point $p$. But, in normal coordinates, these equations are equivalent to

$$\begin{aligned} \nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} &= 0 \\ (g^{-1})^{ab} \nabla_a F_{bc} &= 0. \end{aligned}$$

But now, these equations equate the components of one tensor to another, so the tensors themselves must be equal. Hence we can write them using abstract indices

$$\begin{aligned} \nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} &= 0 \\ \nabla^\mu F_{\mu\nu} &= 0 \end{aligned}$$

In general, to generalise a physical law from special relativity to general relativity, we should

1. express the physical law in terms of tensors in Minkowski space,
2. replace all partial derivatives with covariant derivatives, and
3. replace the Minkowski metric $m$ with the metric $g$.

The principle that all physical laws should be expressed as tensorial equations, encapsured by these rules, is sometimes called the *principle of covariance*.

Note that the equations of motion will couple to the Einstein equations via the energy-momentum tensor – so typically you have to solve the Einstein equations and the matter equations of motion simultaneously.