C7.5 Lecture 14: The Schwarzschild solution 1

The metric and gravitational redshift

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Now we know the Einstein equations, it's time to look at some solutions.

One solution, with $\Lambda = 0$ and $T_{\mu\nu} = 0$, is Minkowski space (in fact, here $R_{\mu\nu\rho\sigma} = 0$).

What other solutions are there? One way to generate these is to solve the initial value problem, but this often does not lead to *explicit* solutions.

An alternative approach is to look for explicit solutions which are highly symmetric. The first success of this approach came just one year after the Einstein equations were published, with the discovery of the *Schwarzschild solution*.

This is a solution to the Einstein vacuum equations without a cosmological constant, i.e. $G_{\mu\nu} = 0$. It is *static*, *spherically symmetric* and *asymptotically flat*. This means that we can write the metric in coordinates (t, r, θ, ϕ), where

1. the components of the metric are independent of t. (*stationarity*).
2. The metric is also invariant under $t \mapsto -t$ (*staticity*).
3. The components of the metric are invariant under a family of transformations that can be parametrised by $SO(3)$ matrices, whose orbits are topological spheres (*spherical symmetry*).
4. As $r \to \infty$, the metric components approach[1] the components of the Minkowski metric written in spherical polar coordinates (*asymptotic flatness*).

[1] Asymptotic flatness actually requires that the metric approaches the Minkowski metric at a certain rate, but there are various possible rates and we will not go into the messy details here.

What is the difference between “stationary” (metric components independent of t) and “static” (stationary, and metric invariant under $t \mapsto -t$)?

Consider the two dimensional metric

$$g = -\mathrm{d}t\mathrm{d}\theta$$

This is stationary but not static.

For a more physical example, consider sphere rotating at a constant speed in an otherwise empty universe. This situation is stationary: it looks the same at all points in time. But if we reverse the direction of time, then the sphere rotates in the opposite direction!

Birkhoff's theorem

A uniqueness/rigidity result:

Birkhoff's theorem

Every spherically symmetric solution to the Einstein vacuum equations is locally isometric to either Schwarzschild spacetime or to Minkowski space.

This theorem is important because it means that the Schwarzschild metric characterises spacetime outside of *any* spherically symmetric matter distribution, regardless of the interior structure of the matter (e.g. the density profile of some fluid). In this case, the metric will *not* agree with the Schwarzschild metric *inside* the matter distribution (where $T_{\mu\nu} \neq 0$), where the metric will generally depend on the specific details of the matter.

The metric

$$g = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)$$

here $M > 0$ is a constant (called the *mass*), and the ranges of the coordinates are as follows:

- $t \in \mathbb{R}$
- $0 < \theta < \pi$
- $0 < \phi < 2\pi$
- For now, $2M < r < \infty$. For example, we might be considering the region outside a spherically symmetric star, where the radius of the star is much larger[2] than $2M$.

[2] Modelling the sun as spherically symmetric, the surface $r = 2M$ is around 1km from its centre – well inside the region where there is matter. For the Earth, this surface is around 1cm from the centre!

Checking that this metric is actually a solution to the Einstein equations is extremely tedious, but it is a calculation that everyone should do at one point in their lives (and then never again!).

There are well-understood degeneracies in the metric at $\theta = 0$, π and also at $\phi = 0$, 2π. These, of course, are the usual coordinate issues with the sphere, and reflect the fact that, technically, we need to use two coordinate charts to cover the whole space. On the other hand, something odd is clearly going on with the metric at $r = 0$ and at $r = 2M$. We'll return to this point later.

Gravitational redshift

Two observers, Alice and Bob, move along integral curves of the vector field ∂_t, at two different radii but with the same angular coordinates. Their worldlines, parametrised by the coordinate t, are

$$\begin{aligned} \text{Alice:} \quad & (t, r_A, \theta_0, \phi_0) \\ \text{Bob:} \quad & (t, r_B, \theta_0, \phi_0) \end{aligned}$$

where θ_0 and ϕ_0 are constants, and $2M < r_A < r_B$.

Alice sends Bob regular signals, using light rays. Light rays travel along null geodesics, which we can also parametrise by the coordinate time t. Since these are null radial lines,

$$\begin{aligned} 0 &= g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}t} \frac{\mathrm{d}x^b}{\mathrm{d}t} \\ &= -\left(1 - \frac{2M}{r}\right) + \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 \\ \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} &= \left(1 - \frac{2M}{r}\right). \end{aligned}$$

Integrating from $r = r_A$ when $t = t_A$, to $r = r_B$ when $t = t_B$, we find

$$\int_{r_A}^{r_B} \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r = t_A - t_B.$$

Importantly, the coordinate time difference $t_A - t_B$ is itself is a constant, *independent of the initial time when the signal was transmitted* (you could just read this off directly from the fact that the metric is stationary). So, if Alice sends repeated signals at (coordinate) time intervals Δt, then they will be received by Bob at (coordinate) time intervals Δt.

Remember that t is just a coordinate, and has no intrinsic meaning. The physical quantity is the proper time.
Along a worldline where r, θ, ϕ are constants, we calculate

$$\begin{aligned} -1 &= g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau} \\ &= -\left(1-\frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \\ \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}\tau} &= \left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}. \end{aligned}$$

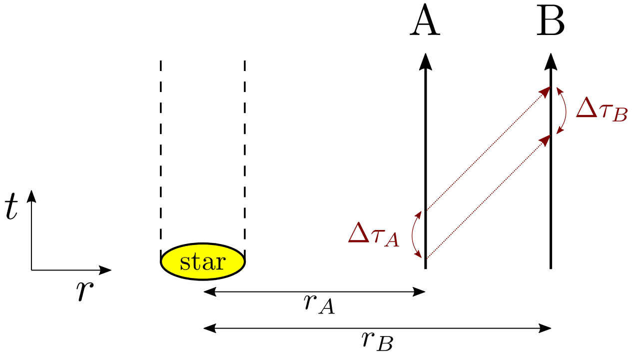
So, along Alice and Bob's worldlines, differences in proper times satisfy

$$\Delta\tau_{A/B} = \left(1-\frac{2M}{r_{A/B}}\right)^{\frac{1}{2}}\Delta t,$$

and the ratio of the emission frequency to the received frequency is

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \frac{\left(1-\frac{2M}{r_B}\right)^{\frac{1}{2}}}{\left(1-\frac{2M}{r_A}\right)^{\frac{1}{2}}}.$$

Since $r_B > r_A$, $\Delta\tau_B > \Delta\tau_A$. So less time passes for Alice than for Bob: Bob receives the signals at a lower frequency than the emitted frequency. *Clocks run slower in a gravitational field.* This is *gravitational redshift*. Note that, as $r_A \to 2M$ the ratio of frequencies tends to infinity – a kind of *infinite redshift*, which we will interpret later.



$\Delta\tau_B > \Delta\tau_A$: Bob receives signals from Alice at a slower rate than they are emitted.