C7.5 Lecture 15: The Schwarzschild solution 2

Geodesics in Schwarzschild

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We'll use a Lagrangian approach to derive the geodesics. Provided we parametrise by an affine parameter, we can take the Lagrangian to be

$$
\begin{aligned}
L &= g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} \\
&= -\left(1 - \frac{2M}{r}\right) \dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2,
\end{aligned}
$$

where the 'dots' represent derivatives with respect to some *affine* parameter λ.

Conserved quantities

With Lagrangians, it's always a good idea to start with conserved quantities.

Since the Lagrangian is independent of t, we have the conserved 'energy'

$$\begin{aligned} E &:= -\frac{1}{2}\frac{\partial L}{\partial \dot{t}} \\ &= \left(1 - \frac{2M}{r}\right)\dot{t}. \end{aligned}$$

Similarly, the Lagrangian is independent of ϕ, so we have the conserved angular momentum about the z axis

$$\begin{aligned}\Omega &:= \frac{1}{2}\frac{\partial L}{\partial \dot{\phi}} \\ &= r^2 \sin^2\theta\dot{\phi}.\end{aligned}$$

The Lagrangian itself is constant: in the timelike case (i.e. for a massive particle) we can choose the affine parameter λ to be the proper time τ, while in the spacelike case we can choose the proper distance s, and so we have

$$-\left(1-\frac{2M}{r}\right)\dot{t}^2+\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2=-K$$

$$=\begin{cases}-1 & \text{(timelike)}\\ 0 & \text{(null)}.\end{cases}$$

Finally, we can use spherical symmetry to rotate the manifold so that the particle moves only in the equatorial plane $\theta = \frac{\pi}{2}$. To be more precise: we can use the $SO(3)$ isometries to rotate so that the particle

- is initially in the equatorial plane,
- has initial velocity tangent to this plane

Then the equation of motion for θ is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(r^2\dot{\theta}\right) - r^2 \sin\theta\cos\theta\dot{\phi}^2 &= 0 \\ \Rightarrow r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\dot{\phi}^2 &= 0. \end{aligned}$$

So, if $\theta\big|_{\lambda=0} = \frac{\pi}{2}$ and $\dot{\theta}\big|_{\lambda=0} = 0$, we have $\ddot{\theta}\big|_{\lambda=0} = 0$. From this it follows that $\theta = \frac{\pi}{2}$ always, since $\theta \equiv \frac{\pi}{2}$ solves the ODE with the correct initial conditions.

Putting this all together, we find the evolution equation for the r coordinate

$$\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2+\frac{\Omega^2}{r^2}+K=\left(1-\frac{2M}{r}\right)^{-1}E^2$$

$$\Rightarrow \frac{1}{2}\dot{r}^2+\frac{\Omega^2}{2r^2}\left(1-\frac{2M}{r}\right)-\frac{MK}{r}=\frac{E^2-K}{2}.$$

This is the equation of motion of a particle with energy $\frac{1}{2}(E^2-K)$, moving in an *effective potential*

$$V(r)=-\frac{MK}{r}+\frac{\Omega^2}{2r^2}-\frac{Mm^2}{r^3}.$$

Timelike geodesics

In this case $K = 1$ and the effective potential is

$$V(r) = -\frac{M}{r} + \frac{\Omega^2}{2r^2} - \frac{M\Omega^2}{r^3}.$$

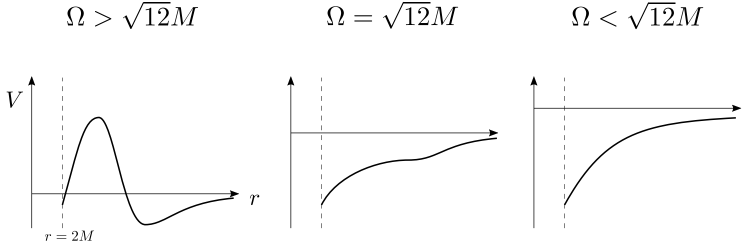
The first term is the Newtonian gravitational potential and the second term is the angular momentum barrier. The third term is new and does not appear in Newtonian theory.

At large r, $V \sim -Mr^{-1}$, and at $r = 2M$, $V = -\frac{1}{2}$ (remember that we are only working in the region $r > 2M$ for now).

The extrema of the potential are at

$$V' = 0 \Rightarrow r = \frac{\Omega^2}{2M}\left(1 \pm \sqrt{1 - 12\frac{M^2}{\Omega^2}}\right)$$

so if $\Omega > \sqrt{12}M$ there are two local extrema, at $\Omega = \sqrt{12}M$ these two extrema collide (leaving a single inflection point), and at $\Omega < \sqrt{12}M$ there are no real extrema.



The effective potential in a Schwarzschild spacetime, for various values of the conserved angular momentum Ω.

Timelike circular orbits

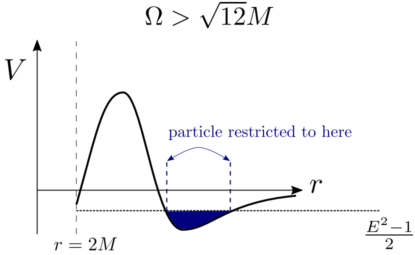
Circular orbits have $\dot{r} = 0$ and $\ddot{r} = 0$ – the latter means that we must be at a local extrema of the effective potential.

Labelling these extrema by r_- and r_+, with $r_+ > r_-$, we find that the extrema at r_- is always *unstable* (i.e. it is a local maximum) while that at $r = r_+$ is *stable*. The *innermost (marginally) stable circular orbit* (ISCO) is obtained when $\Omega = \sqrt{12}M$, when $r = 6M$. The energy of these orbits can be calculated: e.g. for a stable circular orbit,

$$\frac{E^2 - 1}{2} = V(r_+).$$

Bound orbits

If $\Omega > \sqrt{12}M$ then there are bound orbits which are not circular. These have energies satisfying $V(r_+) < \frac{E^2-1}{2} < 0$.



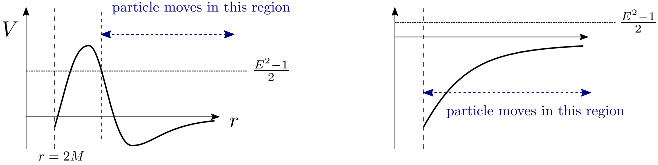
Bound orbits in Schwarzschild.

Unbound orbits

If $\Omega > \sqrt{12}M$ then there are also unbound orbits, which have energies satisfying $E^2 \geq 1$.

For smaller angular momentums ($\Omega \leq \sqrt{12}M$) the situation is interesting: in this regime there is no local maximum of the effective potential. As before, there are unbound orbits (with $E^2 \geq 1$) – but these orbits will only reach infinity if they are outgoing *initially*. All other orbits – that is, orbits with $E^2 < 1$ or with $\dot{r} < 0$ initially – will eventually reach the surface $r = 2M$.

$\Omega > \sqrt{12}M$ $\qquad\qquad$ $\Omega < \sqrt{12}M$



Unbound orbits in Schwarzschild.