C7.5 Lecture 16: The Schwarzschild solution 3

Perihelion precession and the deflection of light

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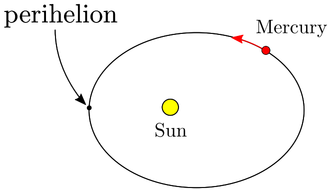
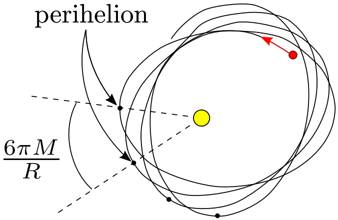
Perihelion precession

A big scientific puzzle before the advent of general relativity was the *anomolous precession of the perihelion of Mercury*. The *perihelion* is the closest point of approach to the sun. In Newtonian theory planets move on ellipses, with the perihelion always occurring at the same point in space. But observations had shown that the perihelion of Mercury is *precessing* – on each orbit, the perihelion occurs at a slightly different angle (see figure 1).

Newtonian orbit



GR orbit



The orbits of the planets (especially Mercury) are close to circular, so our approach to this problem will be to treat Mercury as a point mass travelling on a circular orbit. We will then give this orbit a small perturbation and see what happens.

It is convenient to use the coordinate $u = \frac{M}{r}$ instead of r, and to parametrise the orbits by ϕ instead of the proper time τ. Then we have

$$\frac{\mathrm{d}u}{\mathrm{d}\phi} = \frac{\mathrm{d}u}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^{-1} = -\frac{M}{\Omega}\dot{r}.$$

The radial equation that we derived last time, written in terms of u and $\frac{\mathrm{d}u}{\mathrm{d}\phi}$ instead of r and $\frac{\mathrm{d}r}{\mathrm{d}\tau}$, is

$$\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 - \frac{M^2}{\Omega^2}u + \frac{1}{2}u^2 - u^3 = \frac{M^2(E^2-1)}{2\Omega^2}.$$

Differentiating with respect to ϕ (and assuming $\frac{\mathrm{d}u}{\mathrm{d}\phi} \neq 0$), we obtain

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} - \frac{M^2}{\Omega^2} + u - 3u^2 = 0.$$

Next we set $u = \frac{M}{R} + \epsilon v(\phi)$ and expand in orders of ϵ. Equating the "zero-th order terms" we find $R^2 - \frac{\Omega^2}{M} R - 3\Omega^2 = 0$. Substituting this back into the equation for u and equating terms of order ϵ,

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\phi^2} + \left(1 - 6\frac{M}{R}\right) v = 0.$$

For stable circular orbits $R > 6M$. Then this equation has periodic solutions, with period

$$T_{\text{period}} = \frac{2\pi}{\left(1 - 6\frac{M}{R}\right)^{\frac{1}{2}}} = 2\pi + \frac{6\pi M}{R} + \mathcal{O}\left(\left(\frac{M}{R}\right)^2\right),$$

so the perihelion *precesses* by an additional $6\pi M/R$ per orbit. This matches the observed anomalous precession of Mercury!

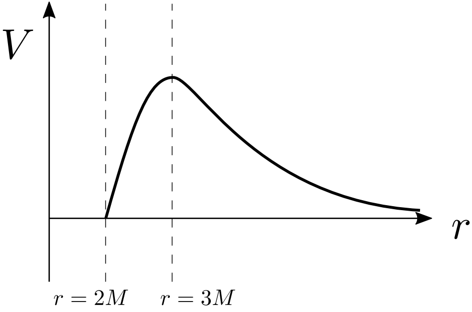
Gravitational bending of light

One of the other classic tests of general relativity is the bending of light when it passes near a massive object.

For this purpose, we need to use the massless geodesic equation rather than the massive one, i.e. we take $K = 0$. The equation for r is then

$$\frac{1}{2}\dot{r}^2 + \frac{\Omega^2}{2r^2}\left(1 - \frac{2M}{r}\right) = \frac{E^2}{2},$$

where 'dots' are derivatives with respect to an affine parameter along the null geodesic (not the proper time!).



The effective potential for null geodesics moving in the Schwarzschild spacetime.

For this effective potential, there is only one local extremum, at $r = 3M$. This is called the *light ring* or *photon sphere* – on this surface, light can orbit the central object (assuming that this central object is smaller than $3M$)! However, unlike the massive case, these orbits are *always* unstable.

For the bending of light, we are interested in unbounded orbits, so the energy satisfies

$$0 < \frac{E^2}{2} < V(3M) = \frac{\Omega^2}{27M^2}.$$

As before, it is useful to set $u = \frac{M}{r}$ and use ϕ as a parameter along the curve instead of the affine parameter. Then we obtain the equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u - 3u^2 = 0.$$

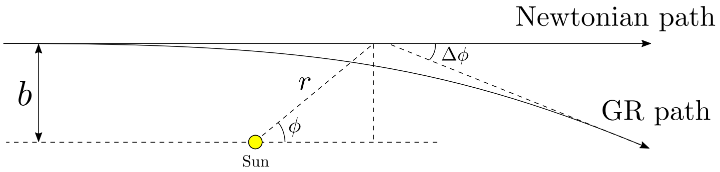
In the Newtonian theory the third term is absent. So, in the Newtonian theory, the solutions are

$$u = \frac{M}{b}\sin(\phi - \phi_0)$$

where b and ϕ_0 are constants. This equation can be rewritten

$$r\sin(\phi - \phi_0) = b.$$

There is no gravitational deflection in the Newtonian theory! The constant b is called the *impact parameter*: in the Newtonian theory, it measures the closest distance of the curve to the origin.



Gravitational deflection of light.

Now, let's reintroduce the quadratic term, which is the correction from GR.

Consider *large* impact parameters (so the light ray passes far from the central region), so we'll set $b = B\epsilon^{-1}$. We'll also set $\phi_0 = 0$ for simplicity. The solution is

$$u = \epsilon \frac{M}{B} \sin\phi + \epsilon^2 v(\phi),$$

and we expand in powers of ϵ:

$$\epsilon^2 \left(\frac{\mathrm{d}^2 v}{\mathrm{d}\phi^2} + v - 3\frac{M^2}{B^2} \sin^2\phi \right) + \mathcal{O}(\epsilon^3) = 0.$$

Ignoring lower order terms, the general solution to this equation is

$$v = \alpha \sin\phi + \beta \cos\phi + \frac{M^2}{B^2}(1 + \cos\phi)^2$$

for some constants α and β. For a particle coming in 'from the left', both the perturbation v and its derivative should vanish when $\phi = \pi$. These conditions give $\alpha = \beta = 0$.

Putting these calculations together, the solution (up to $\mathcal{O}(\epsilon^2)$) is

$$u = \epsilon\frac{M}{B}\sin\phi + \epsilon^2\frac{M^2}{B^2}(1 + \cos\phi)^2 + \mathcal{O}(\epsilon^3).$$

$$u = \epsilon \frac{M}{B} \sin \phi + \epsilon^2 \frac{M^2}{B^2} (1 + \cos \phi)^2 + \mathcal{O}(\epsilon^3).$$

Recall that $r = \frac{M}{u}$. To find the deflection angle, we need to find the value of ϕ such that $u = 0$, with $\phi \leq 0$.

Setting $\phi = -\epsilon(\Delta\phi)$ and expanding in powers of ϵ, we find that

$$0 = -\epsilon^2 \frac{M}{B} (\Delta\phi) + 4\epsilon^2 \frac{M^2}{B^2} + \mathcal{O}(\epsilon^3)$$

$$\Rightarrow (\Delta\phi) = 4\frac{M}{B} + \mathcal{O}(\epsilon)$$

So light rays *are* deflected when they pass near massive objects in general relativity, by an angle of approximately $4\frac{M}{B}$!