C7.5 Lecture 16: The Schwarzschild solution 4

Black holes and singularities

Joe Keir

Joseph.Keir@maths.ox.ac.uk

What happens at $r = 2M$?

So far we have resolutely stuck to the region $r > 2M$. This is fine so long as we are looking at spherically symmetric stars or planets, in which matter modifies the geometry or sufficiently small r. But what if there is no matter there? The Schwarzschild solution is *still* a solution to the *vacuum Einstein equations*, so sooner or later we have to understand what's going on at $r = 2M$.

Recall that the Schwarzschild metric is

$$g = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

There are several places where something goes funny with this expression for the metric. At $r = 2M$ the first component vanishes and the second term becomes infinite. At $r = 0$ all of the components vanish except for the first one, which becomes infinite. Also, at $\theta = 0, \pi$ the $\mathrm{d}\phi^2$ component vanishes. Technically, there is also something slightly unusual going on at $\phi = 0, 2\pi$. Since we are supposed to cover a manifold by open sets which are then mapped onto $\mathbb{R}^n$, these points are not covered by our chart.

The fact that the metric degenerates on the axis $\theta = 0, \pi$ doesn't mean that there is some kind of physical singularity there – after all, exactly the same thing happens in flat space when it's written in spherical polar coordinates!

At the poles, nothing is wrong with the *metric*, but that there is something wrong with the *coordinates*. If we change to different coordinates – for example, changing to polar coordinates with a different pole, or to rectangular coordinates – then these points appear totally normal.

Could something similar be happening at the other places where the metric is problematic, at $r = 2M$ and at $r = 0$?

The Kretschmann tensor

We can obtain hints of what might be going on my looking at *scalar* quantities, since these are independent of the coordinates.

We cannot take contractions of the metric itself: $(g^{-1})^{\mu\nu} g_{\mu\nu} = 4$, which doesn't tell us anything. We also can't construct tensors out of the Christoffel symbols, so our only real choice is the curvature.

The scalar curvature R doesn't work, since the Schwarzschild metric solves the Einstein *vacuum* equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$
$$\Rightarrow R - 2R = 0 \Rightarrow R = 0$$

(hence the Einstein vacuum equations are equivalent to $R_{\mu\nu} = 0$.)

There is a scalar that can be built out of the curvature tensor that can be useful for our purposes, called the *Kretschmann scalar* $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. In the Schwarzschild metric, this has the value

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{12M^2}{r^6}$$

So in a coordinate-independent sense, the curvature is finite at $r = 2M$ but infinite at $r = 0$. This suggests that $r = 2M$ might be an ordinary surface in spacetime – not a singularity – but $r = 0$ might represent a real singularity.

We shouldn't be too hasty to jump to conclusions based on the Kretschmann scalar, for two reasons:

- First, even if the curvature blows up, this does not necessarily mean that spacetime is coming to an end in some kind of singularity (see *weak solutions of PDEs* and *impulsive gravitational waves*.
- There are kinds of "singularities" where the curvature is completely finite, but where the spacetime comes to an end for *global* rather than *local* reasons. These are called *Cauchy horizons*, and they occur inside rotating black holes.

Passing through the event horizon

Returning to the Schwarzschild metric, the Kretschmann scalar *suggests* that the surface $r = 2M$ might be a *coordinate singularity*. To show that this is in fact the case, we need to transform to some different coordinates which 'pass through' the surface $r = 2M$.

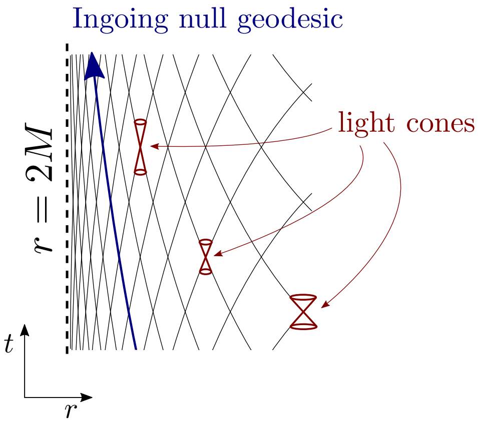
First, we examine the structure of the light cones near $r = 2M$. Since we're in spherical symmetry, we'll look only at the (t, r) plane.
A null vector X in the (t, r) plane satisfies

$$-\left(1 - \frac{2M}{r}\right)(X^t)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(X^r)^2 = 0$$
$$\Rightarrow X^r = \pm\left(1 - \frac{2M}{r}\right)X^t,$$

so a null curve passing through $(t, r) = (t_0, r_0)$ is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\left(1 - \frac{2M}{r}\right)$$
$$\Rightarrow t - t_0 = \pm\left(r - r_0 + 2M\log\left(\frac{r - 2M}{r_0 - 2M}\right)\right).$$

Schwarzschild coordinates (t, r, θ, ϕ)



Light cones and radial light rays in the Schwarzschild spacetime, drawn in "Schwarzschild coordinates" (t, r, θ, ϕ). As $r \to 2M$, the null cones are "squeezed together".

To find coordinates which might allow us to pass through $r = 2M$, we can try to "straighten out" the null cones. This can be achieved by changing from the coordinate t to a coordinate v which is *constant* on the ingoing null curves. Define

$$r^* := r + 2M \log\left(\frac{r - 2M}{2M}\right)$$
$$v := t + r^*.$$

v is constant along ingoing null geodesics (**exercise**). Then

$$\mathrm{d}v = \mathrm{d}t + \mathrm{d}r^* = \mathrm{d}t + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r.$$

Writing the metric in coordinates (v, r, θ, ϕ):

$$g = -\left(1 - \frac{2M}{r}\right) \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right).$$

The metric is no longer diagonal due to the term $2\mathrm{d}v\mathrm{d}r$. The first term still vanishes at $r = 2M$ so we might think that the metric degenerates here. However, the matrix g_{ab} is not degenerate: it is still an invertible matrix, so it has maximal rank. We can check that

$$\begin{aligned} g^{-1} &= (g^{-1})^{ab}\partial_a\partial_b \\ &= 2\partial_v\partial_r + \left(1 - \frac{2M}{r}\right)(\partial_r)^2 + r^{-2}(\partial_\theta)^2 + r^{-2}(\sin\theta)^{-2}(\partial_\phi)^2. \end{aligned}$$

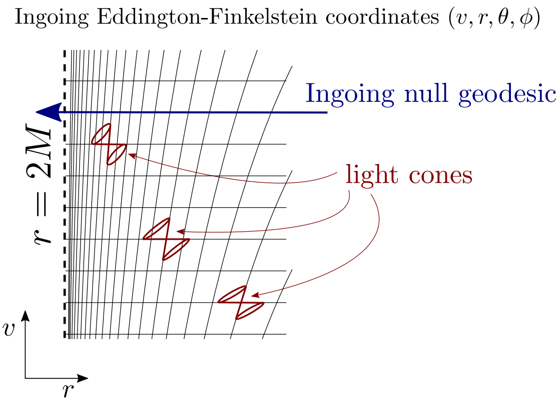
The components of g and g^{-1} are both finite at $r = 2M$ (except for the usual degeneracy from polar coordinates)!

The coordinates (v, r, θ, ϕ) are called *ingoing Eddington-Finkelstein coordinates*[1]. In these coordinates, ingoing null curves are given by $v =$ constant, while outgoing radial null curves (in the region $r > 2M$) are given by

$$\begin{aligned} v - v_0 &= 2\left(r^* - r_0^*\right) \\ &= 2\left(r - r_0 + 2M\log\left(\frac{r - 2M}{r_0 - 2M}\right)\right) \end{aligned}$$

Also, in these coordinates, there is nothing stopping us from considering the 'interior' region $0 < r < 2M$ - the metric is perfectly regular at $r = 2M$. You can also check that the curve given by $r = 2M$ is itself a null curve (**exercise**).

[1] There are also *outgoing Eddington-Finkelstein coordinates.* (u, r, θ, ϕ), where $u = t - r^*$



Light cones and radial light rays in the Schwarzschild spacetime, drawn in ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ). The surface $r = 2M$ is like a one-way membrane – you can enter, but you can never leave! This surface is called the *event horizon*.