C7.5 Lecture 18: The Schwarzschild solution 5

Black holes, white holes, wormholes

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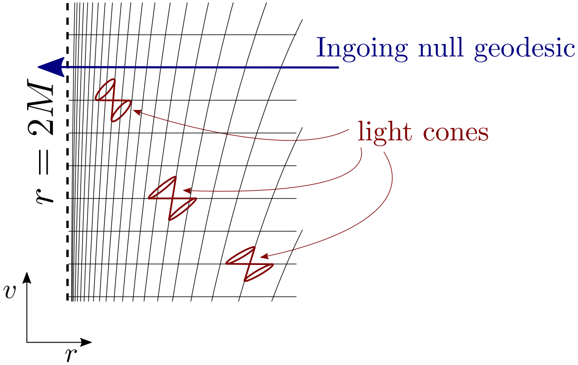
In ingoing Eddington-Finkelstein coordinates, the manifold is completely smooth at $r = 2M$ and we can pass right through the surface $r = 2M$. We find that $r = 2M$ is the event horizon of a black hole: you can pass through it but you can't escape!

Ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ)



Light cones and radial light rays in the Schwarzschild spacetime, drawn in ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ).

We can also consider *outgoing* Eddington-Finkelstein coordinates (u, r, θ, ϕ). In these coordinates, the surface $r = 2M$ can be also be seen to be a null surface, but this time causal curve cross it in the opposite direction: you can pass from the interior $r < 2M$ to the exterior $r > 2M$, but you can't *enter* the interior region! The reason for this apparent discrepancy is that the ingoing and outgoing Eddington-Finkelstein coordinates cover different parts of the manifold (see figure 2). Outgoing Eddington-Finkelstein coordinates cover a *different* 'interior' region $0 < r < 2M$.

To clarify the situation, we can find some coordinates which cover the original region ($r > 2M$) as well as the two extra regions covered by ingoing and outgoing Eddington-Finkelstein coordinates. These are called *Kruskal coordinates* (or *Kruskal-Szekeres coordinates*), and are defined by

$$U = -e^{-\frac{u}{4M}}$$
$$V = e^{\frac{v}{4M}},$$

then in terms of the coordinates (U, V, θ, ϕ) the metric takes the form

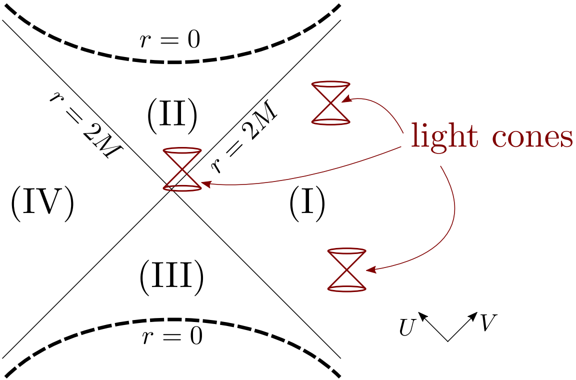
$$g = \frac{32M^3}{r} e^{-\frac{r}{2M}} \mathrm{d}U \mathrm{d}V + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right),$$

where here r is defined implicitly in terms of U and V by the relationship

$$\log(-UV) = \frac{r^*}{2M}.$$

Features of the metric in Kruskal coordinates

- The metric is regular at $r = 2M$.
- $r = 2M$ is given by $UV = 0$.
- The original region is $U < 0$, $V > 0$.
- The region $U > 0$, $V > 0$ is the interior region covered by ingoing Eddington-Finkelstein coordinates.
- The region $U < 0$, $V < 0$ is the interior region covered by outgoing Eddington-Finkelstein coordinates.
- The region $U > 0$, $V < 0$ is a new region – it is actually isomorphic to the starting region!
- The full spacetime, including all four regions, is sometimes called the *maximally extended Schwarzschild spacetime*.



The maximally extended Schwarzschild spacetime. The original Schwarzschild coordinates only cover region (I). Ingoing Eddington-Finkelstein coordinates cover the regions (I) and (II), while outgoing Eddington-Finkelstein coordinates cover regions (I) and (III). Finally, Kruskal-Szekeres coordinates cover regions (I), (II), (III) and (IV).

- Sometimes, instead of U and V, Kruskal coordinates are defined to be the coordinates (T, X, θ, ϕ), where

$$T = \frac{1}{2}(U + V)$$
$$X = \frac{1}{2}(V - U).$$

- $r = 0$ corresponds to the hyperboloid $UV = 1$, although this is technically *not* part of the manifold.
- The original (t, r, θ, ϕ) coordinates are called "Schwarzschild coordinates". In fact, in all four regions (corresponding to different sign choices for U and V) we can define Schwarzschild-type coordinates:
  - the metric takes the same form as the original metric (although in the interior regions, r is a timelike coordinate and t is a spacelike coordinates),
  - *none* of these coordinates extend up to the surface $r = 2M$, i.e. points with $U = 0$ or $V = 0$ are not covered,
  - these are *different coordinates* from the original ones we started with, covering different parts of spacetime!

Interpreting the maximally extended Schwarzschild spacetime

- Throughout the entire spacetime, the Einstein *vacuum* equations $R_{\mu\nu} = 0$ hold. There is no matter present, especially not at $r = 0$, which is not even a part of the manifold!
- The picture of Schwarzschild in Kruskal coordinates is particularly useful because radial light rays travel at 45 degrees on this diagram. Every point on this figure represents a sphere.
- Region (I) is the most familiar region: here $r > 2M$ and, at large distances, the metric approaches the flat Minkowski metric. Worldlines of observers in this region, which always have tangent vectors *inside* the light cones, can always escape to regions of arbitrarily large r.

- Region (II) is the *black hole region*. Once an observer has crossed the surface $r = 2M$ into region (II) they are stuck inside this region. The surface $r = 2M$, called the *event horizon*, acts like a one-way membrane: once you cross $r = 2M$ there is no turning back! On the other hand, there is no local quantity which distinguishes this surface: the curvature is finite, and small observers can cross this surface without noticing anything dramatic.
- Once inside the black hole region r decreases along all worldlines, and in fact *all observers will reach* $r = 0$ *in a finite affine time* (see the example sheet).
- Unlike the surface $r = 2M$, $r = 0$ is a genuine singularity: the curvature is infinite here, and there is no way to extend the manifold[1] beyond $r = 0$.
- Point particles moving on geodesics will not experience any forces, but for a realistic observer of any finite size, tidal forces become infinite as $r \to 0$, ripping the observer apart.

[1] In fact, the manifold cannot be extended in a way such that the metric is even *continuous*, let alone differentiable. This was only proved recently!

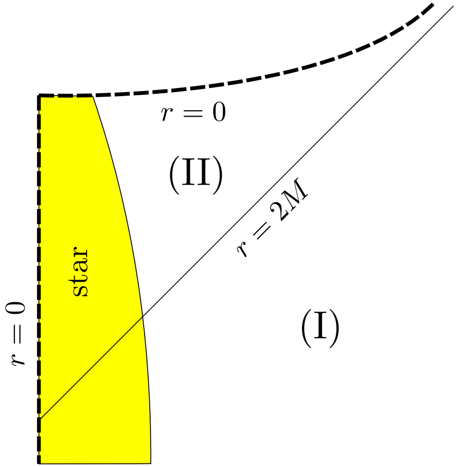
- Because of the singularity at $r = 0$, the Schwarzschild spacetime is *geodesically incomplete*. A *geodesically complete manifold* is one for which all geodesics can be extended arbitrarily far, so that their associated affine parameters can take values in $(-\infty, \infty)$. Minkowski space is geodesically complete, while Schwarzschild is not.

Regions (III) and (IV)

Regions (III) and (IV) are unphysical: in a realistic black hole formed by the collapse of a star they are ‘covered up’ by the matter. However, for the purposes of calculations it is often helpful to work with the full extended Schwarzschild geometry. Since this geometry is invariant under time reversal, these two regions must be present.

Region (III) is called the *white hole* region: it is the time-reversal of the black hole region, and has the time-reversed properties. Observers can leave this region, but they can never enter it!

Region (IV) is a ‘copy’ of the original region (I): it is also asymptotically flat, and looks like Minkowski space far away from the event horizon. It is sometimes said to be ‘another universe’, and inspired ideas about wormholes. However, there is no way for observers to get from region (I) into region (IV), even if it were there!



The spacetime of a "realistic" spherically symmetric collapsing star. By Birkhoff's theorem, the geometry is exactly described by the Schwarzschild metric outside of the star, but inside the star the geometry will be modified, covering up regions (III) and (IV).