C7.5 Lecture 19: Cosmology 1

Homogeneity and isotropy

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What other solutions of the Einstein equations can we obtain by looking for very symmetric solutions? Other than spherical symmetry, what other useful symmetry classes are there?

One important symmetry class is *homogeneous and isotropic* spacetimes. Instead of astrophysical applications, this symmetry class is suitable for studying the entire universe on the largest scales.

Homogeneity

A *homogeneous spacetime* is one where there is a global function τ, called a *time function*, with level sets Σ_τ satisfying:

- The surfaces Σ_τ are spacelike hypersurfaces, i.e. $\mathrm{d}\tau$ is a timelike covector, $g^{-1}(\mathrm{d}\tau, \mathrm{d}\tau) < 0$ (equivalently every curve which lies entirely within Σ_τ is spacelike). By rescaling τ if necessary, we can assume that $g^{-1}(\mathrm{d}\tau, \mathrm{d}\tau) = -1$.
- The surfaces Σ_τ are *homogeneous spaces*. This means that there is a group G which acts on the surface Σ_τ *transitively* (any point can be mapped to any other point by some group element) and by *isometries* (the action of the group preserves the metric). Informally, this says that 'every point looks like every other point' (sometimes called the *Copernican principle*).

Isotropy

The level sets Σ_τ are also required to be *isotropic*. This means that

- for each point $p \in \Sigma_\tau$ and for each pair of unit tangent vector $X, Y \in T_p(\Sigma_\tau)$ (that is, a tangent vector to the submanifold Σ_τ, not a spacetime vector – although such vectors can be considered spacetime vectors that are tangent to the surface Σ_τ), there is an isometry mapping X to Y (see figure 1).

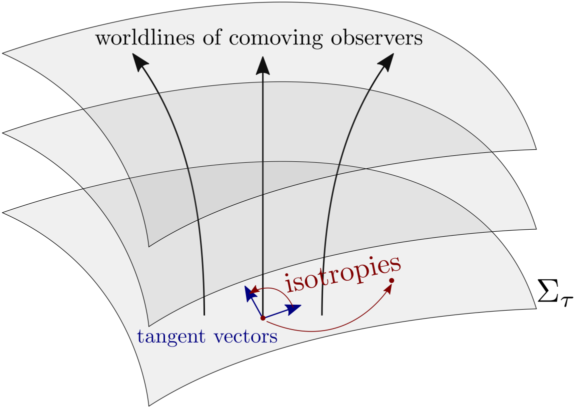
In fact, isotropy implies homogeneity but not vice versa. For example, a torus is homogeneous but not isotropic.

Comoving observers

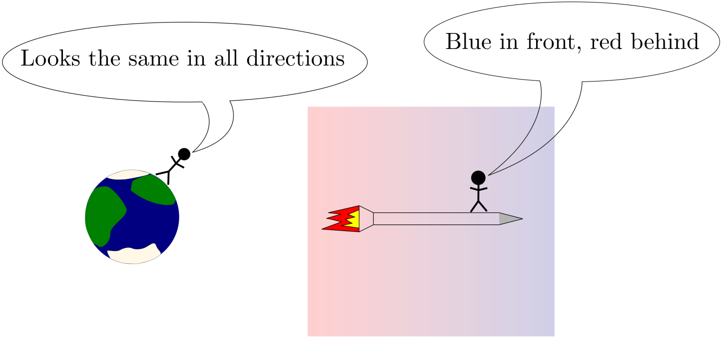
There are special observers, called *isotropic observers* or *comoving observers*, whose worldlines are such that:

- The worldlines are timelike.
- For every pair of unit vectors X, Y which are *orthogonal* to the tangent vector of the wordline, there is an isometry mapping X to Y.

The tangent vector to these worldlines is simply $-(\mathrm{d}\tau)^\sharp$, or, in abstract index notation, $-\partial^\mu\tau$. The $-$ sign is chosen so that these tangents are future-directed.



A homogeneous and isotropic spacetime. There are isometries mapping any point on a surface Σ_τ to any other point on that surface, and also isometries mapping any tangent vector to the surface Σ_τ to any other tangent vector (these isometries are shown in red). Comoving observers move along worldlines which are orthogonal to ths surfaces Σ_τ.



Comoving observer   Non-comoving observer

Comoving and non-comoving observers in a homogeneous and isotropic spacetime.

The geometry of a homogeneous and isotropic universe

Homogeneity and isotropy imply that the spacetime metric can be written as

$$g = -\mathrm{d}\tau^2 + (a(\tau))^2 \underline{g},$$

$\underline{g}$ is the spatial part of the metric, $a(\tau)$ is the *scale factor*. The metric $\underline{g}$ is the metric of a *maximally symmetric space*. There are actually only three options:

- *Flat*: the metric $\underline{g}$ is the Euclidean metric in three dimensions.
- *Closed*: the metric $\underline{g}$ is the standard metric on the 3-sphere $\mathbb{S}^3$.
- *Open*: the metric $\underline{g}$ is a metric of constant *negative* curvature.

In each case there are possible topological complications.

In all cases, the metric can be written in a universal form, called the *Robertson-Walker metric*:

$$\underline{g} = \frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)$$

where $k > 0$ is a closed universe, $k = 0$ is a flat universe and $k < 0$ is an open universe. By rescaling r and the scale factor a, we can always choose $k = 1, 0$ or -1 in the closed, flat or open cases respectively (**exercise**).