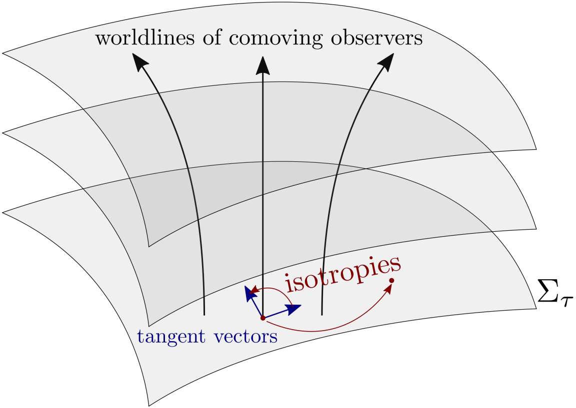
# C7.5 Lecture 20: Cosmology 2

## The Friedmann equations and Hubble's law

Joe Keir

Joseph.Keir@maths.ox.ac.uk




A homogeneous and isotropic spacetime. There are isometries mapping any point on a surface $\Sigma_\tau$ to any other point on that surface, and also isometries mapping any tangent vector to the surface $\Sigma_\tau$ to any other tangent vector (these isometries are shown in red). Comoving observers move along worldlines which are orthogonal to ths surfaces $\Sigma_\tau$.

Tensors and other objects can be decomposed into parts which are tangent to the surfaces $\Sigma_\tau$, and parts which are orthogonal to these surfaces: for example, for a vector $X$ we can write

$$X = X(\tau)(-\mathrm{d}\tau)^\sharp + \underline{X}$$

where $\underline{X}$ is a vector tangent to the surface $\Sigma_\tau$.

This symmetry class makes it easy to calculate the Christoffel symbols and components of the Riemann curvature tensor, for the following reasons:

- Scalar fields on each surface $\Sigma_\tau$ must be constant for consistency with homogeneity. So scalar quantities can only depend on time $\tau$.
- All (co)vector fields tangent to $\Sigma_\tau$ must vanish for consistency with isotropy.
- The only $(1,1)$ tensor fields on the surface $\Sigma_\tau$ consistent with homogeneity and isotropy are those proportional to the Kronecker delta $\delta^i_j$, and the constant of proportionality can only depend on time $\tau$.
- The only $(0,2)$ tensor fields on the surface $\Sigma_\tau$ consistent with homogeneity and isotropy are those proportional to the metric $\underline{g}_{ij}$, and the constant of proportionality can only depend on time $\tau$.
- The only $(2,0)$ tensor fields on the surface $\Sigma_\tau$ consistent with homogeneity and isotropy are those proportional to the inverse metric $(\underline{g}^{-1})^{ij}$, and the constant of proportionality can only depend on time $\tau$.

Recall that we use $i, j, k$ etc. to refer to *spatial* indices. Using these facts together with the form of the metric, some fairly tedious calculations lead to the following expressions for the Ricci tensor components:

$$
\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a} \\
R_{0i} &= 0 \\
R_{ij} &= \left(a\ddot{a} + 2\dot{a}^2 + 2k\right) \underline{g}_{ij},
\end{aligned}
$$

where 'dots' represent derivatives with respect to $\tau$. From these we can calculate

$$
\begin{aligned}
G_{00} + \Lambda g_{00} &= 3\frac{\dot{a}^2 + k}{a^2} - \Lambda \\
G_{ij} + \Lambda g_{ij} &= \left(-2a\ddot{a} - \dot{a}^2 - k + a^2\Lambda\right) \underline{g}_{ij}.
\end{aligned}
$$

The energy momentum tensor must also respect the symmetries imposed by homogeneity and isotropy. This means that we can write

$$T_{00} := \rho$$
$$T_{ij} := pa^2 \underline{g}_{ij}$$

where these equations *define* the 'density' and 'pressure' – we are not necessarily assuming that the matter is a fluid.

# The Friedmann equations

The Einstein equations in a homogeneous and isotropic spacetime are called the *Friedmann equations*. They are the following system of ODEs, which are derived by equating the components of the Einstein tensor and the energy momentum tensor in a homogeneous, isotropic spacetime:

$$3\frac{\dot{a}^2 + k}{a^2} - \Lambda = 8\pi\rho$$
$$2a\ddot{a} + \dot{a}^2 + k - a^2\Lambda = -8\pi p a^2$$

Sometimes the following equation, following from the two above, is useful:

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi(\rho + 3p) + \frac{1}{3}\Lambda$$

To obtain a closed system, we can supplement the Friedmann equations with an equation of state, which expresses the pressure $p$ as a function of the density $\rho$.

By differentiating the first Friedmann equation with respect to $\tau$, we obtain a useful equation for the evolution of the density $\rho$:

$$\begin{aligned} 8\pi\dot{\rho} &= \frac{3\dot{a}}{a^3}\left(2a\ddot{a} - 2\dot{a}^2 - 2k\right) \\ &= \frac{3\dot{a}}{a^3}\left(3\dot{a}^2 - 3k + a^2\Lambda - 8\pi a^2 p\right) \\ &= -24\pi\frac{\dot{a}}{a}(p+\rho) \end{aligned}$$

and so we obtain the equation for the derivative of the density

$$\dot{\rho} = -3\frac{\dot{a}}{a}(p+\rho)$$

# Equations of state

Often equations of state of the following form are considered:

$$p = w\rho,$$

where $w$ is a constant. In this case the density evolves as

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$
$$\Rightarrow \rho \propto a^{-3(1+w)}.$$

Particular values of the constant $w$ have physical meanings:

1. **Dust**: $w = 0$, $\rho \propto a^{-3}$. The pressure vanishes for any value of the density. This is often used to model "ordinary" matter – stars, galaxies, dark matter etc. – since on very large scales there is negligible pressure between these objects. In this case the energy is proportional to the volume element: as the universe expands (and $a$ increases), matter dilutes.
2. **Radiation**: $w = \frac{1}{3}$, $\rho \propto a^{-4}$. From statistical physics, radiation can be treated as a perfect fluid with $p = \frac{1}{3}\rho$. The energy density $\rho$ decreases both due to the increase in the volume element *and* due to redshift of the photons.
3. **Dark energy**: $w = -1$, $\rho =$ const. In this case both the pressure and density are constant, independent $a$. In fact, the energy-momentum tensor is proportional to $g$ (**exercise**). This allows us to reinterpret the cosmological constant $\Lambda$ as a component of "matter", but there is no clear microscopic understanding this. It is usually thought of as the energy of the vacuum, i.e. the ground state energy of the quantum fields describing matter. Cosmological observations give $\Lambda \sim 0$, whereas QFT predicts a value around $10^{120}$ times larger!

# Cosmological redshift and Hubble's law

Alice and Bob are both comoving observers. Alice follows the worldline $(\tau, 0, 0, 0)$ and Bob the worldline $(\tau, r_B, \theta_B, \phi_B)$, where $r_B$, $\theta_B$ and $\phi_B$ are constants. Bob sends light signals at regular intervals $\Delta\tau_B$ to Alice, who receives them at time intervals $\Delta\tau_A$.

Radial null lines in a cosmological spacetime are null geodesics (this follows from isotropy, but **exercise**: check this explicitly). Hence the tangent to an (ingoing) radial null geodesic is

$$X = \partial_\tau - \frac{\sqrt{1 - kr^2}}{a}\partial_r$$

and the path of a null geodesic is $(\tau, r(\tau), \theta_0, \phi_0)$ where

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\frac{\sqrt{1 - kr^2}}{a}.$$

So, if a light ray leaves Bob at time $\tau = \tau_B$ and arrives at Alice at $\tau = \tau_A$, then

$$\int_{r_B}^{0} -\frac{\mathrm{d}r}{\sqrt{1-kr^2}} = \int_{\tau_B}^{\tau_A} \frac{\mathrm{d}\tau}{a(\tau)}.$$

Performing the same calculation for the subsequent signal, and noting that the left hand side is independent of $\tau$, we have

$$\int_{\tau_B+\Delta\tau_B}^{\tau_A+\Delta\tau_A} \frac{\mathrm{d}\tau}{a(\tau)} = \int_{\tau_B}^{\tau_A} \frac{\mathrm{d}\tau}{a(\tau)}.$$

If $\Delta\tau_A$ and $\Delta\tau_B$ are very small compared to $\tau_B - \tau_A$, then to leading order,

$$\frac{\Delta\tau_A}{a(\tau_A)} = \frac{\Delta\tau_B}{a(\tau_B)},$$

so the ratio of the received frequency to the emitted frequency is

$$\frac{\Delta\tau_A}{\Delta\tau_B} = \frac{a(\tau_A)}{a(\tau_B)}.$$

$$\frac{\Delta\tau_A}{\Delta\tau_B} = \frac{a(\tau_A)}{a(\tau_B)}.$$

In an expanding universe, the scale factor $a$ grows over time. Since $\tau_A > \tau_B$, $a(\tau_A) > a(\tau_B)$ and so $\Delta\tau_A > \Delta\tau_B$. So Alice sees the signals at a *lower* frequency than they are emitted by Bob: this is *cosmological redshift*.

Next, suppose that Alice and Bob are close together (relative to the time scale on which $a$ varies). Expanding our expression:

$$\begin{aligned}\frac{\Delta\tau_A}{\Delta\tau_B} &= a(\tau_A)\Big(a(\tau_A) - (\tau_A - \tau_B)\dot{a}(\tau_A)\Big)^{-1} + \mathcal{O}\left((\tau_A - \tau_B)^2\right) \\ &= 1 + (\tau_A - \tau_B)\frac{\dot{a}(\tau_A)}{a(\tau_A)} + \mathcal{O}\left((\tau_A - \tau_B)^2\right) \\ &= 1 + (\tau_A - \tau_B)H(\tau_A) + \mathcal{O}\left((\tau_A - \tau_B)^2\right),\end{aligned}$$

where $H(\tau) := \frac{\dot{a}(\tau)}{a(\tau)}$ is the "Hubble constant". Note that the Hubble constant is not actually constant, but depends on time!

# Hubble's law

$$\frac{\Delta\tau_A}{\Delta\tau_B} \approx 1 + (\tau_A - \tau_B)H$$

is known as *Hubble's law*. The quantity $(\tau_A - \tau_B)$ is a natural measure of the *distance* from Bob to Alice: it is the time taken for light to travel from Bob to Alice (remember that we set $c = 1$).

Hubble's law says that light from nearby galaxies (assumed to move, roughly, along the worldlines of comoving observers) is redshifted in a way which scales *linearly* with the distance to that galaxy, and the constant of proportionality is *Hubble's constant*. Hubble's discovery of this law, with $H > 0$, provided the first clear evidence in favour of an expanding universe.