C7.5 Lecture 21: Cosmology 3

Cosmological models and the big bang

Joe Keir

Joseph.Keir@maths.ox.ac.uk

Recap

$$g = -\mathrm{d}\tau^2 + (a(\tau))^2 \, \underline{g}$$

$$\underline{g} = \frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)$$

$$3\frac{\dot{a}^2 + k}{a^2} - \Lambda = 8\pi\rho$$

$$2a\ddot{a} + \dot{a}^2 + k - a^2\Lambda = -8\pi p a^2$$

Solutions to the Friedmann equations

Next we want to actually solve the Friedmann equations. We have a range of choices to make: the geometry of the spatial slices is determined by the value of k, the matter model is fixed by choosing w, and the cosmological constant Λ also plays a role. This leads to a variety of cosmological models with different behaviours.

The Einstein static universe

Einstein's original model of a static universe was the original motivation for introducing Λ. This is supposed to model the universe now, when the matter is well approximated by 'dust', so we have $w = p = 0$. Substituting into the Friedmann equations and setting $\dot{a} = \ddot{a} = 0$, we obtain

$$\frac{3k}{a^2} - \Lambda = 8\pi\rho$$
$$k - a^2\Lambda = 0$$

from which it follows that

$$k = a^2\Lambda$$
$$\Lambda = 4\pi\rho$$

Since we are dealing with ordinary matter we have $\rho > 0$, so $\Lambda = 4\pi\rho > 0$ and hence (by scaling the coordinates) we can set $k = a^2\Lambda = +1$. This means that the solution is given, in terms of the energy density ρ, as

$$
\begin{aligned}
k &= 1 \\
\Lambda &= 4\pi\rho \\
a &= \frac{1}{\sqrt{4\pi\rho}}.
\end{aligned}
$$

Unfortunately, this solution is dynamically unstable: small perturbations lead to a solution which either rapidly expands or collapses (see example sheet 4).

Matter dominated universes with $\Lambda = 0$

Observations show conclusively that the universe is not static, but expanding, so let's look for such a solution. Again, we'll look for a "matter dominated universe", meaning the matter is well-approximated by dust, and for simplicity we'll take the cosmological constant $\Lambda = 0$. In this case the Friedmann equations are

$$3\frac{\dot{a}^2 + k}{a^2} = 8\pi\rho$$
$$2a\ddot{a} + \dot{a}^2 + k = 0$$

Using this second equation, we find that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(a\dot{a}^2 + ka\right) = \dot{a}\left(2a\ddot{a} + \dot{a}^2 + k\right) = 0$$

and so, if $\dot{a} \neq 0$,

$$\dot{a}^2 = \frac{C}{a} - k$$

where C is constant.

We can identify this constant using the other Friedmann equation:

$$C = \frac{8\pi}{3}\rho a^3$$

Recall that, in the case of dust, $\rho \propto a^{-3}$ so C is indeed a constant.

First, let's consider the flat case $k = 0$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(a^{\frac{3}{2}}) = \frac{3}{2}C^{\frac{1}{2}}$$
$$\Rightarrow a = \left(\frac{3}{2}C^{\frac{1}{2}}\right)^{\frac{2}{3}}\tau^{\frac{2}{3}}$$
$$\rho = \frac{1}{6\pi}\tau^{-2}$$

where we have chosen the solution where $\dot{a} > 0$ and where $a = 0$ at $\tau = 0$. This universe expands from a 'big bang' at $\tau = 0$, where the scale factor goes to zero and the density becomes infinite as τ^{-2}.

The big bang

There is no “place” where the big bang happens: the solution is homogeneous and isotropic at all times. So on each time slice, the universe is the same at every point and in every direction.

As $\tau \to 0$, the scale factor a goes to zero, meaning that the proper distance on a surface Σ_τ between any two comoving observers goes to zero as $\tau \to 0$, and the density $\rho \to \infty$ *everywhere* as $\tau \to 0$.

Nor is there a time ‘before’ the big bang: the spacetime as a whole is a solution to the Einstein equations for all $\tau > 0$, and spacetime terminates in a singularity at $\tau = 0$. This spacetime is geodesically incomplete in the past: there are finite length, inextendible geodesics which terminating at the singularity.

What about closed ($k = 1$) or open ($k = -1$) universes? Let's take $k = 1$ first. Then the Friedmann equations give

$$\dot{a}^2 = \frac{C}{a} - 1.$$

First we substitute $a = Cb^2$. Then we have

$$\frac{b^2}{\sqrt{1-b^2}} \frac{\mathrm{d}b}{\mathrm{d}\tau} = \pm \frac{1}{2C}$$

Now substituting $b = \sin u$ and doing the integral and a bit of algebra, we find

$$C\left(\arcsin\sqrt{\frac{a}{C}} - \sqrt{\frac{a}{C}}\sqrt{1-\frac{a}{C}}\right) = \pm\tau + \text{const.}$$

We can choose the constant to be zero, so that again $a = 0$ when $\tau = 0$. In this case we should also choose the $+$ sign, so that for small positive values of τ, the scale factor a is positive.

For small values of τ, we find (**exercise**) to leading order

$$a = \left(\frac{3}{2}\right)^{\frac{2}{3}} C^{\frac{1}{3}} \tau^{\frac{2}{3}}$$

so that the scale factor approaches zero as $\tau^{\frac{2}{3}}$. As before, $\rho \sim a^{-3}$ so the energy density becomes infinite as τ^{-2}, and the picture of the big bang is similar to before.

There is an important difference in the closed case, however: at the time $\tau = C\pi$ we again have $a = 0$. So, in the closed case, there is a big bang followed by a *big crunch*, when the scale factor decreases to zero in the future and the universe recollapses!

Finally, we consider the open case $k = -1$. Following similar algebra as in the closed case, we obtain the solution

$$C\left(\sqrt{\frac{a}{C}}\sqrt{1+\frac{a}{C}} - \operatorname{arsinh}\sqrt{\frac{a}{C}}\right) = \tau.$$

As $\tau \to 0$ this solution behaves in the same way as the closed and flat cases, but this time, for large values of τ we have $a \sim \tau$, so these universes grow faster than their flat counterparts (remember $a \sim \tau^{\frac{2}{3}}$ in the flat case).

Radiation dominated universes

So far we have only looked at "matter dominated" universes, where the matter content is given by dust. We can also consider the case of radiation, where the pressure is given by $p = \frac{1}{3}\rho$ instead of $p = 0$.
Again, setting $\Lambda = 0$ and setting

$$B = \frac{8\pi}{3} a^4 \rho$$

we find that B is a constant (so $\rho \sim a^{-4}$).
Following similar calculations as in the matter dominated case, the solutions are found to be (**exercise**)

$$k = 0 \qquad \Rightarrow \qquad a = \sqrt{2} B^{\frac{1}{4}} \tau^{\frac{1}{2}}$$

$$k = 1 \qquad \Rightarrow \qquad a = \sqrt{B} \sqrt{1 - \left(1 - \frac{\tau}{\sqrt{B}}\right)^2}$$

$$k = -1 \qquad \Rightarrow \qquad a = \sqrt{B} \sqrt{\left(1 + \frac{\tau}{\sqrt{B}}\right)^2 - 1}$$

Qualitatively these solutions are similar to the matter dominated scenarios, but with different rates.

Dark energy/cosmological constant

Now we'll add back in the cosmological constant Λ, but for simplicity we'll now consider the case in which there is no matter (or, if you prefer, the only matter content of the universe is "dark energy").

In this case, the Friedmann equations are

$$3\frac{\dot{a}^2 + k}{a^2} - \Lambda = 0$$
$$2a\ddot{a} + \dot{a}^2 + k - a^2\Lambda = 0$$

Multiplying the first equation by a^3 and then taking a derivative with respect to τ, we see that (assuming $a > 0$) any smooth solution $a(\tau)$ to the first equation will automatically solve the second equation (**exercise**). So we can concentrate entirely on the first equation.

The solutions in the dark energy case are

$$k = 0 \qquad \Rightarrow \qquad a \propto e^{\pm\sqrt{\frac{\lambda}{3}}\tau}$$

$$k = 1 \qquad \Rightarrow \qquad a = \sqrt{\frac{3}{\Lambda}}\cosh\left(\sqrt{\frac{\Lambda}{3}}\tau\right)$$

$$k = -1 \qquad \Rightarrow \qquad a = \sqrt{\frac{3}{\Lambda}}\sinh\left(\sqrt{\frac{\Lambda}{3}}\tau\right).$$

As before, we appear to have three different solutions. But in fact, all three solutions are different parts of the *same* spacetime – called *de Sitter spacetime*. This spacetime has no 'big bang' – in fact, in the right coordinates, it can be seen to be *static*. The surface $\tau = 0$ in the open case is just a coordinate singularity.

Realistic cosmological models

A realistic model of the universe contains 'dust', 'radiation' and a cosmological constant $\Lambda > 0$ (albeit with Λ *almost* equal to zero!).

Recall the equation that we derived for the evolution of the density:

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

from which it follows that $\rho \propto a^{-3(1+w)}$. Ultimately, this equation is derived from the conservation of the energy momentum tensor for the fluid, $\nabla_\mu T^{\mu\nu} = 0$. If we have several fluids which are *non-interacting*, then each of their energy-momentum tensors is *separately* conserved, and so for *each* fluid f we will have

$$\rho_{(f)} \propto a^{-3(1+w_{(f)})}.$$

The evolution of the universe

Although in general the equations cannot be solved explicitly for a general mixture of this sort, we can still obtain the following picture of what's going on:

1. At early times, when a is very small, the energy density of radiation goes as $\rho_{(\text{rad})} \sim a^{-4}$, and this is the dominant component. So at early times the universe is well approximated by a radiation dominated solution.
2. The energy density of dust behaves as $\rho_{(\text{dust})} \sim a^{-3}$. So as a grows larger, eventually the dust component dominates over the radiation component, and the universe behaves as a matter dominated solution.
3. The energy density of 'dark energy' (or the cosmological constant) is constant, but very small. Eventually, if the universe continues expanding, the scale factor a becomes so large that dark energy dominates over the matter component. At this point, the universe becomes approximately de-Sitter.

Inflation

There is one important addendum to this picture which is often incorporated into modern cosmology, although it remains slightly controversial. To solve certain observational conundrums, cosmologists often suppose that there was an additional time, very early on in the evolution of the universe (before the radiation dominated era) when there was a period of exponential, de Sitter-like growth. This requires some very exotic matter which is able to mimic a large cosmological constant at early times, before falling away so that it is unobservable at the present time.

Course summary

- Developed a geometric understanding of "spacetime".
- In Minkowski space, developed all the tools needed to do physics (worldlines, vectors, tensors, tensor fields. . .)
- Encountered problems with Newtonian gravity and special relativity, including the equivalence principle. Saw hints of spacetime curvature.
- Recovered all of the tools we use to do physics, but now in the context of curved spacetime (including differential calculus).
- Also discovered new phenomena in curved spacetimes: most importantly, the *curvature*.
- Used this to link curvature to matter via the *Einstein equations*.
- Used the Schwarzschild metric to model the spacetime outside a spherically symmetric body, and found several new physical effects (gravitational redshift, perihelion precession, deflection of light).

- Understood the Schwarzschild metric as the metric of a black hole: encountered an *event horizon* and a *singularity*.
- Examined the symmetry class of *homogeneous and isotropic spacetimes*, and applied this to cosmology. Encountered *cosmological redshift*.
- Derived the *Friedmann equations*, which are the Einstein equations in a cosmological spacetime.
- Solved the Friedmann equations, and discovered the *big bang*!

Thank you!