Office hours 2

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Typos in lecture 4.5

In lecture 4.5 (examples), there were a couple of typos in the question about ladders:

1. The parameter λ, used to parametrise the spacelike curve which we think of as "the ladder in its rest frame", was missing a minus sign – I should have defined it so that it increases when we go from the front of the ladder to the back.
2. There was another typo: the formula $\lambda = s\sqrt{1 - v^2}$, which should have been $s = \lambda\sqrt{1 - v^2}$.

Lecture numbering and when to watch them

To give you an idea of the pace of the course, last year:

- Lectures 1-4 covered background material, spacetime, Newtonian gravity, special relativity and the equivalence principle.
- Lectures 5-8 covered most of differential geometry, up to (but not including) curvature.
- Lectures 9-12 covered curvature, the Einstein equations and the exterior Schwarzschild solution, including solar system GR effects.
- Lectures 13-16 covered the Schwarzschild black hole and cosmology .

These line up with the questions on problem sheets 1-4.

Tangent vectors to curves

Given a curve $\gamma : [0,1] \to \mathcal{M}$, we defined the tangent vector to the curve V, at the point $p = \gamma(\lambda_0)$, to be the operator which acts on smooth functions f via

$$V\big|_p(f) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(f \circ \gamma\right)\big|_{\lambda_0}.$$

$V\big|_p(f)$ is not independent of f; it is an operator acting on the function f. What we actually want to show is that equivalence classes of tangent vectors (under the equivalence relation $V_1 \cong V_2$ if $V_1(f) = V_2(f)$ for all smooth f) forms an N-dimensional vector space.

To show this, let $V \in [V]$ be a tangent vector at the point $p \in \mathcal{M}$, tangent to a curve γ. Then

$$V(f) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(f \circ \phi_U^{-1} \circ \phi_U \circ \gamma\right)\big|_{\lambda_0},$$

now,

$$f \circ \phi_U^{-1} = \tilde{f} : \mathbb{R}^n \to \mathbb{R}, \qquad \text{and}$$
$$\phi_U \circ \gamma = (x^a(\lambda)) : [0,1] \to \mathbb{R}^n,$$

so

$$V(f) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\tilde{f}(x^a(\lambda))\right)\big|_{\lambda_0} = \left.\frac{\mathrm{d}x^a(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda_0} \left.\frac{\partial \tilde{f}}{\partial x^a}\right|_{x^a(\lambda_0)} = V^a \left.\frac{\partial \tilde{f}}{\partial x^a}\right|_{x^a(\lambda_0)}.$$

We have seen that, if we choose some chart ϕ_U covering the point p (i.e. some local coordinates), then for any tangent vector V we can write

$$V(f) = V^a \frac{\partial \tilde{f}}{\partial x^a}\bigg|_{x^a(\lambda_0)}.$$

We can use this to give a map $\Phi : T_p(\mathcal{M}) \to \mathbb{R}^N$, via

$$\begin{aligned} V \in [V] &\in T_p(\mathcal{M}) \\ \Phi([V]) &= V^a. \end{aligned}$$

This map is well-defined: let $W \in [V]$ be some tangent vector to another curve γ', in the same equivalence class. Then, since $V(f) = W(f)$, we easily see that $V^a = W^a$.

We want to show that this map is a bijection, i.e. that Φ^{-1} exists. Let $V^a \in \mathbb{R}^N$; then consider the set of curves in $\phi_U(U) \subset \mathbb{R}^N$,

$$[\gamma] := \left\{ \gamma^a : [0,1] \to \mathbb{R}^N \, , \, \gamma^a(\lambda_0) = \phi_U(p) \, , \, \frac{\mathrm{d}}{\mathrm{d}\lambda}\gamma^a \bigg|_{\lambda_0} = V^a \right\} .$$

We can "lift" this to a set of curves on the manifold:

$$[\hat{\gamma}] = \left\{ \phi_U^{-1} \circ \gamma \, , \, \gamma \in [\gamma] \right\} .$$

Then we can easily check that $\Phi^{-1} : V^a \to [\hat{\gamma}]$ is the inverse of Φ.

Note that this map depends on the choice of local coordinates.

The “transformation law” for scalars

In the notes it’s stated that scalar fields don’t transform when we change coordinates, and also that we can form a scalar field by contracting a vector and a covector. How can we see this?

A scalar field is a map $f : \mathcal{M} \to \mathbb{R}$. Local coordinates are maps $\phi_U : \mathcal{M} \to \mathbb{R}^N$. Since f is not a function of ϕ_U, if we change from ϕ_U to ϕ_V, the function f doesn't change!

However, the function $\tilde{f}_U = f \circ \phi_U^{-1}$ *does* change. This is a function from $\mathbb{R}^N \to \mathbb{R}$; it is "$f$ as a function of the local coordinates". Then we can see that

$$\tilde{f}_V = \tilde{f}_U \circ \phi_U \circ \phi_V^{-1} = \tilde{f}_U \circ \phi_{V,U},$$

where $\phi_{V,U}$ is a transition function. This just means that

$$\tilde{f}_V(y) = \tilde{f}_U(x(y)).$$

A covector $\eta : T_p(\mathcal{M}) \to \mathbb{R}$, so if $X \in T_p(\mathcal{M})$, then $\eta(X) : p \to \mathbb{R}$ is just a scalar function at p. Hence it is independent of a change of local coordinates.

We can also see this directly using the fact that $\eta(X) = \eta_a X^a$, together with the definition of the components:

$$\eta_a = \eta(\partial_a)$$

$$X^a = X(x^a).$$

From these we can deduce the transformation laws for the components, and then we find that

$$(\eta')_{a'}(X')^{a'} = \eta_a \left(\frac{\partial x^a}{\partial y^{a'}}\right) \left(\frac{\partial y^{a'}}{\partial x^b}\right) X^b = \eta_a \delta^a_b X^b = \eta_a X^a,$$

using the inverse function theorem.

Tensors

Two (equivalent) definitions:

1. A tensor at a point p is an element of the (vector space/tensor) product of the tangent space at p, to some power n, times the cotangent space at p, to some power m.
2. A tensor at a point p is a multilinear map from the (vector space/tensor) product of the tangent space at p, to some power m, times the cotangent space at p, to some power n, to the reals.

The second is easier to work with in practice.

E.g. to check that some map T from $T_p(\mathcal{M})^2 \to \mathbb{R}$ defines a tensor, we need to check that

$$\begin{aligned} & T(aX + X', bY + Y') \\ &= abT(X, Y) + aT(X, Y') + bT(X', Y) + T(X', Y') \end{aligned}$$

for all constants a and b, and for all vectors X and Y.

If we're talking about tensor fields instead of tensors, then you just need to check the same thing, except that now you need to check C-infinity linearity, i.e. you need to check that the same equation holds, but now a and b are allowed to be scalar fields (not just constants) and X and Y are allowed to be vector fields (not just vectors at p).

There is a difference between a tensor (which is a multilinear map) and the *components* of a tensor (which is an array of numbers) – just like the difference between a vector and the components of a vector.

We *derived* expressions for how the components of a tensor transform when we change coordinates. Hence one way to show that an object is *not* a tensor is to show that its components don't transform in the required way.

On the other hand, given an array of numbers T_{ab}, we can form a tensor

$$T = T_{ab} dx^a dx^b,$$

then the components of T, with respect to the coordinates x^a, are T_{ab}.

One unfortunate piece of notation when writing tensors: suppose we have a $(1, 1)$ tensor Q. Then we can write this as

$$Q = Q_a{}^b \mathrm{d}x^a \partial_b,$$

but here you might think that the covector $\mathrm{d}x^a$ is acting on the vector ∂_b. If you prefer, you can write this as

$$Q = Q_a{}^b \mathrm{d}x^a \otimes \partial_b.$$

In the notes I don't use the symbol $\otimes$, but I do try to use brackets to indicate "acting on", e.g. ηX is just the tensor product of η and X, while $\eta(X)$ is η acting on X.