Office hours 3

Joe Keir

Joseph.Keir@maths.ox.ac.uk

Symmetric products and notation for the metric

The metric g can be expanded in terms of coordinate differentials as

$$g = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = g_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b.$$

We can check that g_{ab} are the components of g:

$$\begin{aligned} g(\partial_a, \partial_b) &= g_{cd}[\mathrm{d}x^c(\partial_a)][\mathrm{d}x^d(\partial_b)] \\ &= g_{cd}\delta^c_a\delta^d_b = g_{ab}. \end{aligned}$$

Because the components of the metric are symmetric $g_{ab} = g_{ba}$, we don't need to worry about symmetrising over the differentials.

On the other hand, in an expression like

$$g = \mathrm{d}x^1\mathrm{d}x^2 = \mathrm{d}x^1 \otimes_S \mathrm{d}x^2,$$

we must assume that we are using the "symmetric tensor product":

$$\mathrm{d}x^1 \otimes_S \mathrm{d}x^2 = \frac{1}{2}\mathrm{d}x^1 \otimes \mathrm{d}x^2 + \frac{1}{2}\mathrm{d}x^2 \otimes \mathrm{d}x^1,$$

and so the components of g are $g_{12} = g_{21} = \frac{1}{2}$.

Often we use letters rather than numbers to label coordinates, e.g.

$$g = \mathrm{d}x\mathrm{d}y = \mathrm{d}x \otimes_S \mathrm{d}y,$$

and then we might write $g_{xy} = g_{yx} = \frac{1}{2}$.

"Sliding" vectors around

Why do we say that there is no natural way to "slide vectors around" in a manifold? Mathematically, given a vector $V_1 \in T_{p_1}\mathcal{M}$, why is there not a "corresponding" vector $V_2 \in T_{p_2}\mathcal{M}$?

Consider a curved surface $S \subset \mathbb{E}^3$. "Vectors" tangent to this surface are just vectors in $\mathbb{E}^3$, tangent to the surface in the ordinary sense.

If we try to slide this vector around, staying parallel to itself using the ambient $\mathbb{E}^3$ structure, then in general a vector at p_1 will not be tangent to S at p_2, and so cannot be interpreted as a "vector" at p_2.

If, as we slide the vector around, we alternate between 'sliding' a distance ϵ, and projecting onto the surface, then this is the same as *parallel transport*[1]. But in this case, the resulting vector at p_2 turns out to depend on the path taken from p_1 to p_2 (unless S is flat).

[1] Using the Levi-Civita connection associated with the metric that S inherits from the ambient space.

Spherical polar coordinates

Why don't the ordinary polar coordinates (θ, ϕ) cover the *entire* sphere $\mathbb{S}^2$?

- $[0, \pi] \times [0, 2\pi)$ is not open in $\mathbb{R}^2$!
- This map is not a bijection: all points $(0, \phi)$ for $\phi \in [0, 2\pi)$ are mapped to the same point (the north pole) on the sphere, and similarly for all points (π, ϕ).

Composing a curve and a chart

We often write a curve "in local coordinates", or we write it as $\gamma^a(\lambda)$ or $x^a(\lambda)$. What exactly does this mean?

Consider $\tilde{\gamma} = \phi_U \circ \gamma : [0,1] \to \mathbb{R}^n$. A point in $\mathbb{R}^n$ is of the form $(x^0, x^1, \ldots, x^{n-1})$. Hence

$$\tilde{\gamma}(\lambda) = (x^0(\lambda), x^1(\lambda), \ldots, x^{n-1}(\lambda)) = (\gamma^0(\lambda), \gamma^1(\lambda), \ldots, \gamma^{n-1}(\lambda)).$$

These are the $x^a(\lambda)$ or the $\gamma^a(\lambda)$.

Abstract vs. concrete indices

What exactly is "wrong" with an expression like $T^{\mu\mu}$ or T^{aa}? The answer depends on whether we are working with abstract indices or concrete ones.

Working in abstract indices, the indices only tell us what kind of object we're dealing with. Repeated indices indicate a "trace", which we know is well defined independent of the coordinate system. E.g.

$$T^{\mu}{}_{\mu} = \sum T(f^a, e_a),$$

where e_a are a basis for the tangent space, and f^a are the dual basis for the cotangent space, so $f^a(e_b) = \delta^a_b$. This is independent of the choice of basis e_a.

We might think that $T^{\mu\mu}$ indicates the object whose value, in any coordinate system, is $\sum_a T^{aa}$. But the problem is that this object is not a scalar field – its value depends on the coordinate system chosen. So it doesn't make sense to talk about $T^{\mu\mu}$ without specifying a coordinate system.

Working in concrete indices, there is nothing especially wrong with considering the quantity

$$T^{aa} = \sum_a T^{aa}.$$

However, for the same reasons as before, we cannot regard this object as defining a scalar field.

In other words, this equation does define a scalar f, but that scalar will depend on which coordinate system we are working in. So if we change coordinates, then in general we will find that

$$f \neq (T')^{a'a'}.$$

Vectors vs. covectors and their actions

A vector V can act on scalars f to give $V(f)$, while a covector $\mathrm{d}f$ can act on vectors V to give the same quantity, $V(f)$. So what's going on?

First, let's clarify how a vector $V \in T_p(\mathcal{M})$ acts on scalar fields f. Recall that a vector V is an equivalence class of tangent vectors $[V]$ to curves $[\gamma]$. We first choose any one of these curves $\gamma \in [\gamma]$ through the point p. Then

$$V(f) = \frac{\mathrm{d}}{\mathrm{d}\lambda} (f \circ \gamma)(\lambda)\Big|_{\lambda=\gamma^{-1}(p)}.$$

Now, when a vector acts on a scalar field, it *does not* act in a C^∞ linear manner. Instead, it obeys the Leibniz rule. So, if we compute $V(ab)$ for two scalar fields a and b, we find that

$$V(ab) = aV(b) + bV(a).$$

On the other hand, the covector field $\mathrm{d}f$ acts on vector fields V via

$$\mathrm{d}f(V) = V(f).$$

This *does* act in a C^∞ linear manner:

$$\mathrm{d}f(aV + V') = (aV + V')(f) = aV(f) + V'(f).$$