Office hours 4

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Acceleration of Jacobi fields

In the lecture notes I wrote an equation of the form

$$\frac{\partial^2}{\partial^2 \tau} J = \nabla_X \nabla_X J,$$

and we later went on to show that this is equal to $R(X, J)X$. But it's not at all clear what I mean by the partial derivatives of the vector field J – in fact, I only really wrote this expression to suggest that we should think of $\nabla_X \nabla_X J$ as being like the acceleration of J along the geodesic with tangent vector X.

This can be contrasted with the case of a scalar field: for any scalar field f, we have

$$\frac{\partial f}{\partial \tau} = X(f) = \nabla_X f,$$

and similarly for higher derivatives.

An obvious way to interpret the expression $\frac{\partial^2}{\partial\tau^2} J$ is as the vector field whose components are $\frac{\partial^2}{\partial\tau^2} J^a$. But this depends on which coordinates we choose – it's not a coordinate-independent expression.

In fact, if we use coordinates (τ, s, x^2, x^3), where x^2 and x^3 are chosen so that the surface generated by the timelike geodesics is given by $x^2 = x^3 = 0$ (so these coordinates are "transverse" to this surface), then we find that

$$J = \frac{\partial}{\partial s},$$

so the components of J are $(0, 1, 0, 0)$. Hence the derivatives of the components of J just vanish, if we use these coordinates!

There are some “coordinates” in which the derivatives of the components of J are the same things as the components of the vector $\nabla_X \nabla_X J$. In fact, what we need are not coordinates but a “frame”.

A frame is a set of vector fields e_a, such that, at every point where the frame is defined, these vectors span the tangent space. An example of a frame is the set of coordinate induced vector fields ∂_a – but not all frames have to be of this form.

We set up our frame as follows: at the point $\tau = 0$, along each geodesic, we choose some frame. We then transport this frame along the integral curves of X by parallel transport: that is, our frame vector field satisfy

$$\nabla_X e_a = 0$$

for all a. If we want, we can choose $e_0 = X$, since that $\nabla_X X = 0$.

These e_a span the tangent space initially (i.e. at $\tau = 0$). In fact, they always span the tangent space, since their inner products with one another don't change as we move along the geodesics:

$$\frac{\partial}{\partial \tau} g(e_a, e_b) = X(g(e_a, e_b)) = g(\nabla_X e_a, e_b) + g(e_a, \nabla_X e_b) = 0,$$

and, if the matrix $g_{ab} = g(e_a, e_b)$ is non-degenerate, then the vectors $\{e_a\}$ span the tangent space.

Since the e_a span the tangent space at each point along the integral curves of X, we can expand the vector J in terms of this basis:

$$J = J^a e_a.$$

The scalars J^a are the components of J relative to the frame e_a.

Now let's calculate the components of the vector $\nabla_X \nabla_X J$:

$$(\nabla_X \nabla_X J)^a e_a = \nabla_X \nabla_X J = \nabla_X \nabla_X (J^a e_a) = (\nabla_X \nabla_X J^a)\, e_a = \frac{\partial^2 J^a}{\partial \tau^2} e_a.$$

Hence, relative to this frame, *the components of the vector* $\nabla_X \nabla_X J$ *are the same as the second derivatives of the components of* J.

Frames vs coordinates

Given some coordinates x^a, we can always find a frame of coordinate induced vector fields, i.e. we can choose $e_a = \partial_a$. However, the converse is not true: given a frame e_a, there *does not* always exist coordinates such that $e_a = \partial_a$. How can we see this?

If $e_a = \partial_a$, then we can easily see that $[e_a, e_b] = 0$ for all pairs of basis vectors. However, for a general frame, there is no reason why the commutator must vanish. This actually leads to the correct converse: *if* $\{e_a\}$ *are frame such that* $[e_a, e_b] = 0$*, then locally there exist coordinates such that* $e_a = \partial_a$ (Frobenius' theorem).

A perfect frame?

Wouldn't it be great if we could find a frame $\{e_a\}$ with the property that each frame vector field is parallel transported along all of the other frame vector fields? That is, where

$$\nabla_{e_a} e_b = \nabla_a e_b = 0 \quad \text{for all } a, b.$$

If we could find such a frame, then taking covariant derivatives would be easy:

$$\begin{aligned}\nabla_X Y = \nabla_X(Y^a e_a) &= X(Y^a)e_a + Y^a \nabla_X e_a = X(Y^a)e_a + Y^a \nabla_{X^b e_b} e_a \\ &= X(Y^a)e_a + Y^a X^b \nabla_b e_a = X(Y^a)e_a,\end{aligned}$$

so $(\nabla_X Y)^a = X(Y^a)$: the components of the covariant derivative of Y would be the same as the derivatives of the components of Y.

We will show that it is **impossible** to find such a basis, unless the manifold is flat!

First choose a point p, and at p pick two of our basis vectors e_a and e_b. Construct the affinely parametrised geodesics through p with tangent vectors at p given by e_a and e_b.

Along the geodesic with initial tangent e_a, let's parallel transport both the vectors e_a and e_b, and let's call the resulting vectors $e_{(a,1)}$ and $e_{(b,1)}$. So these satisfy

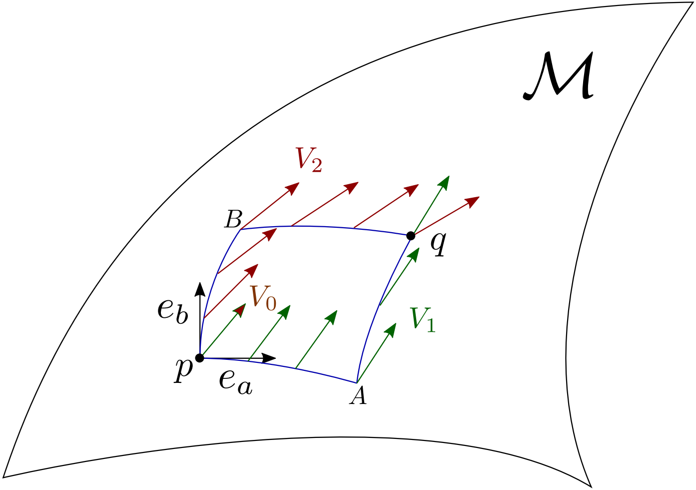
$$\nabla_{e_{(a,1)}} e_{(a,1)} = 0 \qquad \nabla_{e_{(a,1)}} e_{(b,1)} = 0$$
$$e_{(a,1)}\big|_p = e_a \qquad e_{(b,1)}\big|_p = e_b.$$

Let's do the same thing along the geodesic with initial tangent e_b, forming the vectors $e_{(a,2)}$ and $e_{(b,2)}$:

$$\nabla_{e_{(b,2)}} e_{(b,2)} = 0 \qquad \nabla_{e_{(b,2)}} e_{(a,2)} = 0$$
$$e_{(b,2)}\big|_p = e_b \qquad e_{(a,2)}\big|_p = e_a.$$

Now, after moving an affine parameter length ϵ along the geodesic with initial tangent e_a, let's shoot off a geodesic in the direction of $e_{(b,1)}$. Similarly, after moving an affine parameter length ϵ along the other geodesic, let's shoot off a geodesic in the direction of $e_{(a,2)}$. These two geodesics will cross at a point q, forming a small "geodesic rectangle".

Now let's take some arbitrary vector V_0 at p. We can parallel transport this in two ways to the point q around the edges of the rectangle. Let's call the vector that we get by initially moving in the direction of e_a V_1, and let's call the vector that we get by moving initially in the e_b direction V_2. Our goal is to calculate $V_1 - V_2$ at the point q.



$\mathcal{M}$
V_2
B
q
e_b
V_0
V_1
p
e_a
A

To make things as easy as possible, we'll compute the components of V with respect to normal coordinates at p, and we'll only do our calculations to leading order in ϵ.

First we calculate the components of V_1 at the point A. Along the geodesic joining p to A, the components satisfy

$$0 = (\nabla_{e_{(a,1)}} V_1)^c = \frac{\mathrm{d}}{\mathrm{d}\lambda}(V_1)^c + \Gamma^c_{de}(e_{(a,1)})^d (V_1)^c.$$

Since we move a parameter distance ϵ, and since the Christoffel symbols are smooth functions vanishing at p, we find that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(V_1)^c = \mathcal{O}(\epsilon)$$

$$\Rightarrow (V_1)^c\big|_A = (V_1)^c\big|_p + \mathcal{O}(\epsilon^2) = (V_0)^c + \mathcal{O}(\epsilon^2).$$

Next let's compute the components of V_1 at the point q. Along the curve joining A to q we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(V_1)^c + \Gamma^c_{de}(e_{(b,2)})^d (V_1)^e = 0.$$

Now, we can see that

$$\Gamma^c_{de} = \Gamma^c_{de}\big|_A + \mathcal{O}(\epsilon^2)$$

$$(e_{(b,2)})^d = (e_b)^d + \mathcal{O}(\epsilon).$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(V_1)^c + \Gamma^c_{de}\big|_A (e_b)^d (V_1)^e = \mathcal{O}(\epsilon^2).$$

and moreover, the parameter distance from A to q is $\epsilon + \mathcal{O}(\epsilon^2)$.

Integrating this equation from A to Q, we find

$$(V_1)^c\big|_q = (V_1)^c\big|_A - \epsilon\Gamma^c_{de}\big|_A(e_b)^d(V_1)^e\big|_A + \mathcal{O}(\epsilon^3)$$

$$= (V_0)^c - \epsilon\Gamma^c_{de}\big|_A(e_b)^d(V_0)^e + \mathcal{O}(\epsilon^3).$$

Using the fundamental theorem of calculus, integrating along the geodesic from p to A we also have

$$\Gamma^c_{de}\big|_A = \Gamma^c_{de}\big|_p + \int_0^\epsilon \frac{\mathrm{d}}{\mathrm{d}\lambda}\Gamma^c_{de}\mathrm{d}\lambda$$

$$= \int_0^\epsilon (e_{(a,1)})^f\partial_f\Gamma^c_{de}\mathrm{d}\lambda$$

$$= \epsilon(e_a)^f\partial_f\Gamma^c_{de}\big|_p + \mathcal{O}(\epsilon^2).$$

Putting this all together, we find that

$$(V_1)^c\big|_q = (V_0)^c - \epsilon^2(\partial_f \Gamma^c_{de})\big|_p (e_a)^f (e_b)^d (V_0)^e + \mathcal{O}(\epsilon^3).$$

Repeating everything for V_2, we find that

$$(V_1)^c\big|_q - (V_2)^c\big|_q = \epsilon^2(\partial_d \Gamma^c_{fe} - \partial_f \Gamma^c_{de})\big|_p (e_a)^f (e_b)^d (V_0)^e + \mathcal{O}(\epsilon^3).$$

Now, remember that we are working in normal coordinates at p. Hence these derivatives of the Christoffel symbols are the components of the Riemann tensor:

$$(V_1)^c\big|_q - (V_2)^c\big|_q = \epsilon^2 R^c{}_{edf}\big|_p (e_a)^f (e_b)^d (V_0)^e + \mathcal{O}(\epsilon^3)$$

$$= \epsilon^2 \left(R(e_b, e_a) V_0\right)^c + \mathcal{O}(\epsilon^3).$$

If the curvature at p is nonzero, then the vectors V_1 and V_2 will generally differ at q.

What does this have to do with our original problem? Remember that we wanted a frame $\{e_a\}$ which satisfies $\nabla_a e_b = 0$ for all a and b. The problem is that we can apply what we've just learned to one of these vectors, e.g. e_c. To find e_c at q, we can either parallel-transport it along the integral curves of e_a and then the integral curves of e_b, or vice-versa. However, this will yield different results for the vector e_c at q, unless the curvature vanishes!

Tensors vs linear maps

What exactly is the relationship between a linear map (like $R(X, Y)$) and a tensor (like the Riemann tensor R)?

To simplify things, let T be a rank $(1, 1)$ tensor at p (easy to generalise to other rank tensors). Then we can construct a linear map as follows:

$$\tilde{T} : T_p(\mathcal{M}) \to T_p(\mathcal{M})$$
$$\eta(\tilde{T}(X)) = T(X, \eta) \quad \text{for all covectors } \eta.$$

In abstract indices:

$$\left(\tilde{T}(X)\right)^\mu = T_\nu{}^\mu X^\nu$$

Conversely, let $\tilde{T} : T_p(\mathcal{M}) \to T_p(\mathcal{M})$ be a linear map. Then we can use this to define a $(1,1)$ tensor at p:

$$T(X, \eta) = \eta(\tilde{T}(X)).$$

(This is just like what we did with the Riemann tensor).

Technically the tensor T and the linear map $\tilde{T}$ are different objects, but they have the same "components" if we define the components of a linear map in the obvious way: with respect to a frame e_a, with dual f^a,

$$\left(\tilde{T}(X)\right)^a = \tilde{T}_b{}^a X^b$$

$$\Rightarrow \tilde{T}_b{}^a = f^a(\tilde{T}(e_b)) = T(e_b, f^a) = T_b{}^a.$$