Office hours 5

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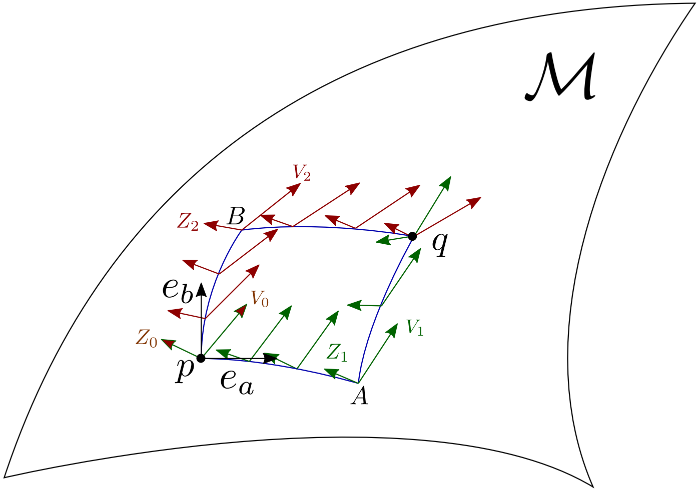


$\mathcal{M}$
V_2
Z_2
B
q
e_b
V_0
V_1
Z_0
Z_1
p
e_a
A

$$
\begin{aligned}
(V_1)^c\big|_q - (V_2)^c\big|_q &= \epsilon^2 R^c{}_{edf}\big|_p (e_a)^f (e_b)^d (V_0)^e + \mathcal{O}(\epsilon^3) \\
&= \epsilon^2 \left(R(e_b, e_a)V_0\right)^c + \mathcal{O}(\epsilon^3).
\end{aligned}
$$

Take another vector Z_0 at p, and transport it in the same way as the vector V_0. Then we can take the inner products of $(V_1 - V_2)$ with either Z_1 or Z_2 at p.

$$
Z_1^\flat(V_1 - V_2) = g(V_1 - V_2, Z_1) = \epsilon^2 R(Z_0^\flat, V_0, e_b, e_a) + \mathcal{O}(\epsilon^3).
$$

For a metric-compatible connection, the inner product $g(V, Z)$ is constant as we parallel-transport things around the rectangle, so $g(V_1, Z_1) = g(V_0, Z_0) = g(V_2, Z_2)$. Hence

$$
\begin{aligned}
g(V_1 - V_2, Z_1) &= g(V_1, Z_1) - g(V_2, Z_1) \\
&= g(V_2, Z_2) - g(V_2, Z_1) = g(V_2, Z_2 - Z_1),
\end{aligned}
$$

so $R(Z_0^\flat, V_0, e_b, e_a) = -R(V_0^\flat, Z_0, e_b, e_a) + \mathcal{O}(\epsilon)$.

Torsion

Now consider a metric with nonvanishing torsion. We can no longer work in normal coordinates at p, so the Christoffel symbols at p don't vanish.

Let's work in coordinates where the point p is at the origin, and do calculations to order ϵ^2. Moving a parameter distance ϵ along a geodesic in the X^a direction, we arrive at the point with coordinates $x^a = A^a$, where

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\lambda^2} = -\Gamma^a_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} \frac{\mathrm{d}x^c}{\mathrm{d}\lambda}$$

$$x^a(0) = 0 \qquad \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}(0) = X^a,$$

so we see that $A^a = \epsilon X^a - \epsilon^2 \Gamma^a_{bc}\big|_p X^b X^c + \mathcal{O}(\epsilon^3)$.
Parallel transporting the vector Y along this integral curve we form the vector Y_1,

$$\frac{\mathrm{d}(Y_1)^a}{\mathrm{d}\lambda} = -\Gamma^a_{bc}(Y_1)^b (X_1)^c.$$

So Y_1 at the point A has components

$$(Y_1)^a = Y^a - \epsilon\Gamma^a_{bc}\big|_p Y^b X^c + \mathcal{O}(\epsilon^2).$$

Hence the point p_1 has coordinates

$$(p_1)^a = \epsilon(X^a + Y^a) - \epsilon^2\Gamma^a_{bc}\big|_p Y^b X^c + \mathcal{O}(\epsilon^3).$$

Following first the integral curve of the vector Y_2 and then X_2, we find that

$$(p_2)^a = \epsilon(X^a + Y^a) - \epsilon^2\Gamma^a_{bc}\big|_p Y^b X^c + \mathcal{O}(\epsilon^3),$$

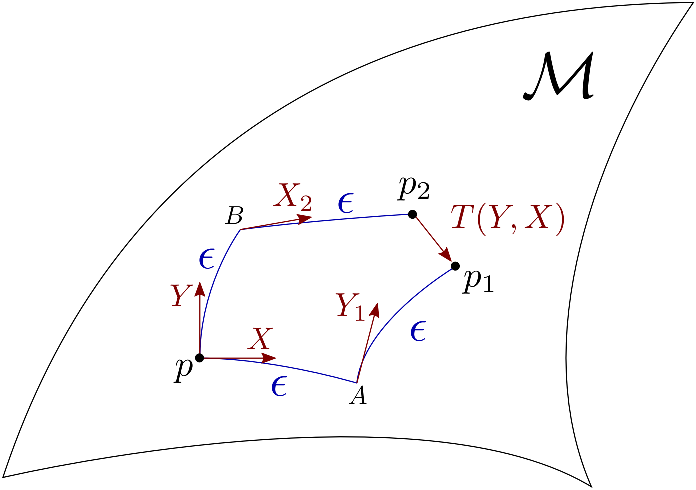
and so

$$(p_1)^a - (p_2)^a = \epsilon^2 \left(\Gamma^a_{bc}\big|_p - \Gamma^a_{cb}\big|_p\right) Y^b X^c + \mathcal{O}(\epsilon^3)$$

$$= \epsilon^2 T^a_{bc}\big|_p Y^b X^c + \mathcal{O}(\epsilon^3)$$

$$= \epsilon^2 \left(T(Y,X)\right)^a + \mathcal{O}(\epsilon^3).$$

This gives an interpretation of the torsion tensor: if we travel a distance ϵ in the X direction and then in the Y direction then we reach the point p_1, while travelling first in the Y direction and then the X directions takes us to the point p_2. The vector $T(Y, X)$ points from p_2 to p_1: following a geodesic in the direction of this vector for a unit distance will take us (to leading order in ϵ) from p_2 to p_1.


$\mathcal{M}$
B
X_2
ϵ
p_2
$T(Y,X)$
ϵ
p_1
Y
Y_1
ϵ
X
p
ϵ
A

A calculation

$$
\begin{aligned}
X^a Y^b [\nabla_a, \nabla_b] Z^c &= X^a Y^b \left(\nabla_a \nabla_b - \nabla_b \nabla_a\right) Z^c \\
&= X^a \nabla_a (Y^b \nabla_b Z^c) - (X^a \nabla_a Y^b) \nabla_b Z^c \\
&\quad - Y^b \nabla_b (X^a \nabla_a Z^c) + (Y^b \nabla_b X^a) \nabla_a Z^c \\
&= \nabla_X \nabla_Y Z^c - \nabla_Y \nabla_X Z^c \\
&\quad - (\nabla_X Y - \nabla_Y X)^a \nabla_a Z^c \\
&= \left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right)^c
\end{aligned}
$$

Generalities about Lagrangians

Given a Lagrangian $L(\boldsymbol{x}, \dot{\boldsymbol{x}}, \lambda)$, where $\dot{x}^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}$, the Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0.$$

Now consider a different Lagrangian $\tilde{L} = f(L)$. The Euler-Lagrange equations for $\tilde{L}$ are

$$0 = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(f' \frac{\partial L}{\partial \dot{x}^a}\right) - f' \frac{\partial L}{\partial x^a}$$

$$= f'\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a}\right) + f''\left(\frac{\partial L}{\partial \dot{x}^a}\right)\left(\frac{\mathrm{d}L}{\mathrm{d}\lambda}\right).$$

So, if $f' \neq 0$ and $\frac{\mathrm{d}L}{\mathrm{d}\lambda} = 0$, then both L and $\tilde{L}$ lead to the same Euler-Lagrange equations.

Next consider a Lagrangian which is independent of λ. Then the associated *Hamiltonian* is constant:

$$H := \dot{x}^a \frac{\partial L}{\partial \dot{x}^a} - L$$

$$\frac{\mathrm{d}H}{\mathrm{d}\lambda} = \ddot{x}^a \left(\frac{\partial L}{\partial \dot{x}^a}\right) + \dot{x}^a \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\mathrm{d}L}{\mathrm{d}\lambda}$$

$$= \ddot{x}^a \left(\frac{\partial L}{\partial \dot{x}^a}\right) + \dot{x}^a \left(\frac{\partial L}{\partial x^a}\right) - \left(\frac{\partial L}{\partial \lambda} + \dot{x}^a \left(\frac{\partial L}{\partial x^a}\right) + \ddot{x}^a \left(\frac{\partial L}{\partial \dot{x}^a}\right)\right)$$

$$= -\frac{\partial L}{\partial \lambda}.$$

Lagrangians for geodesics

- Choosing $L = \sqrt{|g_{ab}(x)\dot{x}^a\dot{x}^b|}$, this Lagrangian is *reparametrisation invariant*: if $x(\lambda)$ extremises the action, then so does $x(\lambda')$, where λ' is any monotonic function of λ. This can be shown either by
  - checking that the Euler-Lagrange equations are equivalent, or
  - checking directly that the action is invariant.
- We can choose any parameter λ which we like; an affine parameter is any choice such that L is constant.
- Choosing an affine parameter, we can instead use the Lagrangian $L' = g_{ab}(x)\dot{x}^a\dot{x}^b$. This gives the same Euler-Lagrange equations, but now the parameter must be an affine parameter.
- The Hamiltonian associated with L' is $H = L'$, so L' is constant.

Solving the Einstein equations coupled to matter

Consider a scalar field ϕ solving $\nabla^\mu \nabla_\mu \phi = 0$, with energy-momentum tensor

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi.$$

Then to solve "the Einstein equations coupled to matter" we need to solve the system of equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$\nabla^\mu \nabla_\mu \phi = 0.$$

To solve these equations we need to give initial data for the scalar field: ϕ "at time zero" and the normal derivative of ϕ "at time zero". We also need to give something like "the metric and time zero" and the "time derivative of the metric at time zero".