Office hours 7

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Is there always a singularity inside a black hole?

A black hole region is defined as a region of spacetime from which no physical signals can escape to infinity. Does this necessarily mean that there is a singularity?

In a sense, yes: in the black hole region, there will generally be *trapped surfaces*: 2D surfaces whose area decreases when you drag the surface along in both the "outgoing" and "ingoing" null directions. The event horizon itself is *marginally trapped*: when you move along outgoing radial null geodesics its area remains constant ($4\pi(2M)^2$), while the area decreases if you drag it along ingoing null geodesics.

Penrose's famous *singularity theorem* (or *incompleteness theorem*) states that, if you have a trapped surface, then the resulting spacetime (formed by evolving initial data) is geodesically incomplete.

However, *geodesic incompleteness* is not the same as a singularity in the ordinary sense of the word. One way in which a solution to the Einstein equations (taken to mean the maximal development of some initial data) can be geodesically incomplete is to have a singularity which is "strong enough' to prevent us from continuing to solve the Einstein equations.

But this is not the only way – it can be that, instead of *existence* of a larger manifold failing (because of a singularity), *uniqueness* of a larger manifold fails. This is actually what happens inside (perfectly angularly symmetric, asymptotically flat etc. etc.) black holes. The Kretschmann scalar remains bounded, and tidal forces are finite as you approach the "boundary" of spacetime.

Penrose's *strong cosmic censorship conjecture* states that *generic* black holes, it is existence rather than uniqueness which fails, and so there is a "proper" singularity.

All of this may be somewhat academic when it comes to “real” black holes: we expect quantum gravity to be important in regions where the curvature is large, and many physicists expect that quantum gravity will somehow remove singularities.

But we don’t know quantum gravity, and these classical arguments might be important anyway in showing that the curvature *does* grow large.

Matter vs. mass

In relativity, we don't really distinguish between *energy* and *mass* (remember $E = mc^2$). In GR, for general asymptotically flat spacetimes, we can determine the *total energy/mass of the spacetime*. This is the constant M in the Schwarzschild metric. This includes contributions from all of the different types of matter that are present, as well as the energy of the gravitational field. In the case of Schwarzschild, no matter is present, so all of the energy comes from the gravitational field.

Matter, on the other hand, is all of the stuff which contributes to the energy-momentum tensor – it could be electromagnetic fields, fluids, dust etc. If there is some matter present in the solution then it will have some mass, but the mass is only partially contributed by the matter itself: some of the "mass" will be due to energy in the gravitational field.

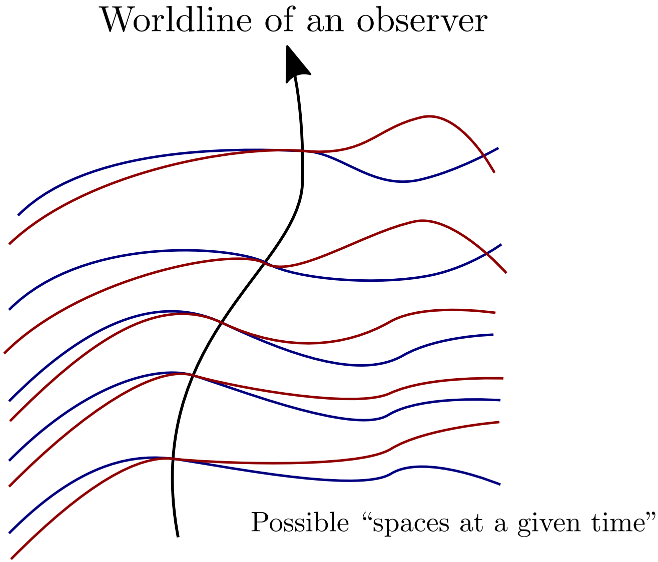
What is “space at a given time”?

There is no unique definition – any “spacelike hypersurface” will do (a spacelike hypersurface is a surface with a timelike normal. Locally, these surfaces can be realised as the level sets of functions t, with $\mathrm{d}t$ a timelike covector field).

If you think of yourself as “at rest”, then it is natural to choose a hypersurface orthogonal to the tangent vector of your worldline (i.e. if the tangent to your worldline is X, then $X \propto -(\mathrm{d}t)^\sharp$), then locally this will look like “space at a given time” for an inertial observer in Minkowski space.

In a homogeneous and isotropic spacetime, there are special spatial surfaces – namely, the surfaces which are homogeneous and isotropic spaces! Then it is natural to use these surfaces.

An observer only “sees” things travelling to their worldline along null or timelike lines, i.e. they see the interior of their past null cones. However, they will think of these signals as having been sent at some point in the past.


Worldline of an observer
Possible “spaces at a given time”

What is this group action and how does it work?

Let G be a group that acts on the manifold: for every $p \in \mathcal{M}$ and every $x, y \in G$, we have

$$\phi_x(p) \in \mathcal{M}$$

$$\phi_x(\phi_y(p)) = \phi_{x \cdot y}(p),$$

and we also require $\phi_e(p) = p$.
For homogeneous and isotropic spacetimes, the group action also maps the level sets of τ to themselves: if $\tau|_p = \tau_1$, then $\tau|_{x(p)} = \tau_1$. So it can also be thought of as a group action on the spatial slices, considered as submanifolds.

The group action is by isometries: informally, it "preserves the metric" or is a "symmetry of the manifold". This means that, for all points $p \in \mathcal{M}$, all vectors X, $Y \in T_p(\mathcal{M})$, and all group elements $x \in G$,

$$g(X, Y) = g((\phi_x)_* X, \phi_x)_* Y).$$

What is $(\phi_x)_* X$? It's the *pushforward* of the vector X by the map ϕ_x. You can think of it as "the vector in $T_{\phi_x(p)}(\mathcal{M})$ which corresponds ot the vector $X \in T_p(\mathcal{M})$".

How do you find this "corresponding vector"? First, take a curve γ through p, with tangent vector (in the equivalence class) X at p. Then we can compose this with ϕ_x to get a curve $\phi_x \circ \gamma$, through $\phi_x(p)$. The tangent vector of this curve at $\phi_x(p)$ is (in the equivalence class) $(\phi_x)_* X$!

Informally, when we use G to move the points on the manifold around, and we also move the vectors around in a natural way, then the metric doesn't change.

Comoving observers

For every pair of unit vectors orthogonal to the worldline of a comoving observer, there is an isometry mapping one vector to the other.

In most cases, the only isometries guaranteed to us are those given by the homogeneous and isotropic conditions, and these only give us isometries of the spatial slices – hence, in general, the "vectors orthogonal to the worldline" must also lie tangent to the spatial slices. But this means that these vectors are also orthogonal to $(\mathrm{d}\tau)^\sharp$:

$$g\left((\mathrm{d}\tau)^\sharp, X\right) = (\mathrm{d}\tau)(X) = X(\tau) = 0,$$

because X is tangent to the level sets of τ. So the tangent to the worldline of a comoving observer must, in general, be proportional to $(\mathrm{d}\tau)^\sharp$ (and actually equal to $-(\mathrm{d}\tau)^\sharp$, with the standard normalisation conventions).

Why do I only say *in general*? Because there are some spaces with extra symmetries: isometries which rotate vectors that do not lie tangent to one of the spatial slices.

For example, Minkowski space is homogeneous and isotropic, and we can use the standard time coordinate t as a time function. However, *any* inertial observer is an isotropic observer (we can perform rotations relative to "their" spatial slices), not just the ones moving along integral curves of $-(\mathrm{d}t)^\sharp = \partial_t$.

What does "future-pointing" mean?

We always assume that our manifolds are equipped with a *time-orientation*: a smooth function τ with $\mathrm{d}\tau$ a nonzero, timelike covector field. Then we say X is future-directed if $X(\tau) > 0$, or equivalently if $g(X, -(\mathrm{d}\tau)^\sharp) < 0$ (note that $-(\mathrm{d}\tau)^\sharp$ is itself future-directed).

If we work with spacetimes constructed by starting with initial data, and then solving the Einstein equations as evolution equations, then we get such a function "for free". In cosmological spacetimes, we also have such a function for free. However, there are spacetimes where such functions cannot be defined! If we lived in such a spacetime, then you could go around some path in spacetime and return to Earth, only to find yourself travelling the opposite direction in time to everyone else. . .

Isotropy implies homogeneity

How do we show that isotropy implies homogeneity? Recall that *isotropy* means that $\exists$ an isometry mapping any unit vector to any other unit vector, while *homogeneity* means that $\exists$ an isometry mapping any point to any other point.

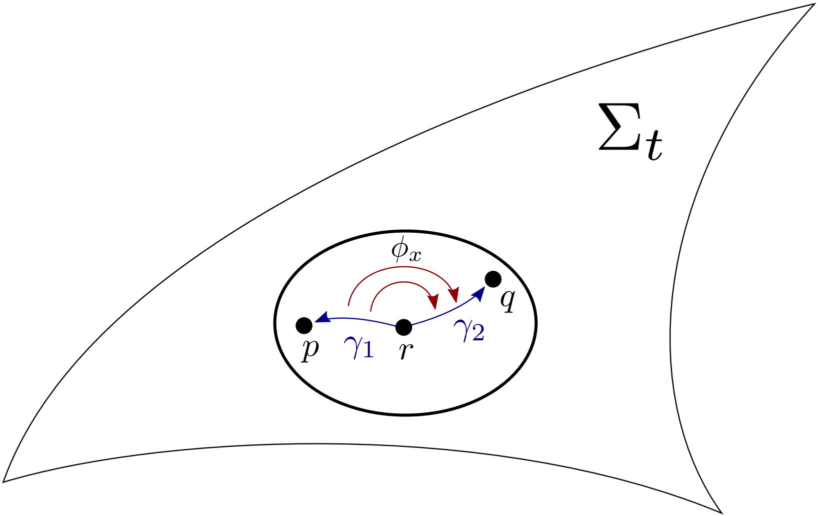
First, consider two points p and q which are close together; we'll show that there's an isometry mapping p to q. We first choose a point r which is equidistant between p and q (recall that we are working on a spatial slice, which is a Riemannian manifold):

$$\inf_{\{\gamma:\gamma(0)=r,\gamma(1)=p\}} \int_0^1 \sqrt{g(\dot{\gamma},\dot{\gamma})}\mathrm{d}\lambda = \inf_{\{\gamma:\gamma(0)=r,\gamma(1)=q\}} \int_0^1 \sqrt{g(\dot{\gamma},\dot{\gamma})}\mathrm{d}\lambda.$$

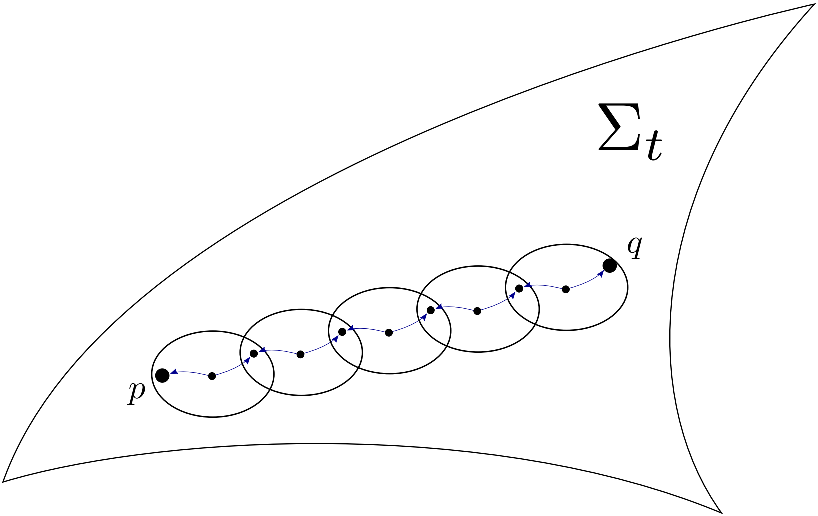
Now consider the geodesics beginning from r: γ_1 joins r to p, and γ_2 joins r to q (these can be shown to exist if p, q and r are sufficiently close together).

By isotropy, there is an isometry mapping $\dot{\gamma}_1\big|_r$ to $\dot{\gamma}_2\big|_r$. Since this is an isometry, it also maps the entire geodesic γ_1 to the geodesic γ_2 (because geodesics depend only on the metric, which is invariant). Moreover, it maps points the point which is a proper distance d along the geodesic γ_1, to the point which is a proper distance d along γ_2 (because proper distance only depends on the metric, which is invariant). Hence it maps p to q.

To "globalise" this argument, we simply "chain together" many of these local isometries.



Σ_t
ϕ_x
q
γ_2
p
γ_1
r


Σ_t
q
p

More on homogeneity and isotropy

Suppose the group G acts on our manifold by isometries. Let $p \in \mathcal{M}$, and consider the *stabilizer of* p, that is, the set of group elements x with $\phi_x(p) = p$. This is a subgroup of $\mathcal{G}$.

By isotropy, the stabilizer subgroup H acts on the tangent space to the spatial slices at p by "isometries" (now we are talking vector space isometries carried out by a linear map, where the vector space $T_p(\Sigma_t)$ is equipped with the metric $(g|_{\Sigma_t})|_p$ and the linear map is the "differential" of ϕ_x). Moreover, H acts transitively on this space, so (assuming the action is also *effective*) $H \cong SO(3)$.

Now choose an arbitrary point $p \in \mathcal{M}$. Since G acts transitively on $\mathcal{M}$, we can identify p with the identity $e \in G$, and then for every other point $q \in \mathcal{M}$, we can identify q with some $x \in G$ where $\phi_x(p) = q$.

In fact, there are multiple group elements which would lead to the same q (and group elements which fix the point p), corresponding to the stabilizer subgroup H. So in fact, points on the manifold can be identified with cosets xH, i.e. $\mathcal{M}$ can be identified with the group quotient G/H.

In general, you can form a homogeneous and isotropic space in n dimensions by taking a Lie group G of dimension $n(n+1)/2$ with a subgroup isomorphic to $SO(n)$, and then considering the space of cosets $G/SO(n)$. For example, $\mathbb{S}^3 \sim SO(4)/SO(3)$.