Numerical Linear Algebra

Sheet 2 — MT20

LU, QR, linear systems and least-squares

This sheet is due 9am two weekdays before the class (Thursday if class is on Monday). Please submit a single pdf file, not separate ones.

1. Show that computing an LU factorization of $A \in \mathbb{R}^{n\times n}$ requires $\frac{2}{3}n^3 + O(n^2)$ operations. Show that the leading term $\frac{2}{3}n^3$ does not change if pivots are used.

2. Compute the leading term in the operation count of QR factorisation of an $m \times n (m \geq n)$ matrix via Householder QR. Do the same for Gram-Schmidt.

3. Show that a nonsingular $A \in \mathbb{R}^{n\times n}$ has a pivoted LU factorisation, i.e., there exists a permutation matrix P s.t. $PA = LU$.

4. (a) Show that the LU factorisation $A = LU$ of a nonsingular $A \in \mathbb{R}^{n\times n}$, if it exists, is unique if one imposes that L's diagonal entries are all 1. (FYI $A = LU$ exists if the leading pricipal submatrices $A(1 : k, 1 : k)$ are nonsingular for all k)

   (b) Show that the Cholesky factorisation $A = R^T R$ of a symmetric positive definite $A \in \mathbb{R}^{n\times n}$ is unique if we impose that R has positive diagonal entries.

   (c) Show that the QR factorisation $A = QR \in \mathbb{R}^{m\times n}$ of a full rank matrix is unique if we impose that R has positive diagonal entries.

5. Suppose $A \in \mathbb{R}^{n\times n}$ is tridiagonal ($A_{ij} = 0$ if $|i - j| > 1$), and it has an LU factorisation $A = LU$ without pivots.

   Show that L, U are both bidiagonal ($L_{ij} = U_{ji} = 0$ if $i > j + 1$).

   Hence show that a linear system $Ax = b$ can be solved in $O(n)$ operations.

   What if QR is used to solve $Ax = b$? Comment briefly.

6. If $A\,x = b$ and $(A + \delta\, A)\,(x + \delta\, x) = b$ show that

$$\frac{\|\delta\, x\|}{\|x + \delta\, x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta\, A\|}{\|A\|}.$$

7. For the linear least squares problem: $\min_x \|A\,x - b\|_2$, show that the solution obtained by QR factorisation of A is the same as the vector y obtained by solving the (square) linear system of equations $A^{\mathrm{T}}\,A\,y = A^{\mathrm{T}}\,b$ (the "normal equations").

   Can you devise a way of solving the linear least squares problem by employing the SVD?

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8. (Computational) By using a built-in code or your own implementation of LU (with pivots) and QR

   (a) Solve $Ax = b$ for an $n \times n$ random matrix (`randn(n)` in matlab) and random vector b. Take $n = k \times 1000$ for $k = 1, 2, \ldots,$ (as large as feasible) and compare the running time and residual $\|Ax - b\|/\|b\|$.

   (b) Examine the stability of QR and LU. For the same A, compute $\|A - QR\|_2/\|A\|_2$ and $\|PA - LU\|_2/\|A\|_2$, and report your findings. (if possible, also try LU without pivots $\|A - LU\|_2/\|A\|_2$)

9. (Optional) Let $A \in \mathbb{R}^{n\times n}$ be partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where $A_{11} \in \mathbb{R}^{k\times k}$ is nonsingular.

   (a) Verify that

$$\begin{bmatrix} I_k & \\ -A_{21}A_{11}^{-1} & I_{n-k} \end{bmatrix} A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

   The $(2, 2)$-block $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is called the *Schur complement* of A_{11} in A.

   (b) Consider running k steps of the algorithm for LU factorisation (suppose this can be done without pivots). Show that the remaining $(n - k) \times (n - k)$ 'active matrix' is equal to the Schur complement $A_{22} - A_{21}A_{11}^{-1}A_{12}$.

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