LU decomposition

$\boxed{A}x = b$ $\quad x = A^{-1}b$ bad idea in practice!

Let $A \in \mathbb{R}^{n\times n}$. Suppose we can decompose (or factorise)

$$PA = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} = \begin{bmatrix} * & & & & \\ * & * & & & \\ * & * & * & & \\ * & * & * & * & \\ * & * & * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{bmatrix} = LU$$

$x = A\backslash b$ (great)

L: lower triangular, U: upper triangular. How to find L, U?

$L_n \cdots L_1 A \to U \iff A = L^{-1} U$

$L^{-1} = $ (low. tri.)

LU decomposition

Let $A \in \mathbb{R}^{n\times n}$. Suppose we can decompose (or factorise)

$$A = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} = \underbrace{\begin{bmatrix} * & & & & \\ * & * & & & \\ * & * & * & & \\ * & * & * & * & \\ * & * & * & * & * \end{bmatrix}}\underbrace{\begin{bmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{bmatrix}} = LU$$

L: lower triangular, U: upper triangular. How to find L, U?

equal to A's.

$$A = \begin{bmatrix} * \\ * \\ * \\ * \\ * \end{bmatrix} \begin{bmatrix} * & * & * & * & * \end{bmatrix} + \begin{bmatrix} & & & & \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \end{bmatrix}$$

L_1 U_1

$$= \underbrace{\begin{bmatrix} * \\ * \\ * \\ * \\ * \end{bmatrix} \begin{bmatrix} * & * & * & * & * \end{bmatrix}}_{L_1U_1} + \underbrace{\begin{bmatrix} 0 \\ * \\ * \\ * \\ * \end{bmatrix} \begin{bmatrix} 0 & * & * & * & * \end{bmatrix}}_{L_2U_2} + \begin{bmatrix} & & & & \\ & & & & \\ & & * & * & * \\ & & * & * & * \\ & & * & * & * \end{bmatrix} = \cdots$$

$L_n U_n$

LU decomposition cont'd

First step:

$$A = \underbrace{\begin{bmatrix} * \\ * \\ * \\ * \\ * \end{bmatrix} \begin{bmatrix} * & * & * & * & * \end{bmatrix}}_{L_1 U_1} + \begin{bmatrix} & & & & \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \end{bmatrix}$$

algorithm:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & & & & \\ A_{31} & & & & \\ A_{41} & & & & \\ A_{51} & & & & \end{bmatrix} = \begin{bmatrix} L_{11} & & & & \\ L_{21} & & & & \\ L_{31} & & & & \\ L_{41} & & & & \\ L_{51} & & & & \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} & U_{15} \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} + \begin{bmatrix} & & & & \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & & & & \\ A_{21}/a & & & & \\ A_{31}/a & & & & \\ A_{41}/a & & & & \\ A_{51}/a & & & & \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}}_{=L_1 U_1} + \begin{bmatrix} & & & & \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \end{bmatrix}$$

$a = A_{11}$

LU decomposition cont'd 2

$$A = \begin{bmatrix} * \\ * \\ * \\ * \\ * \end{bmatrix} \begin{bmatrix} * & * & * & * & * \end{bmatrix} + \begin{bmatrix} 0 \\ * \\ * \\ * \\ * \end{bmatrix} \begin{bmatrix} 0 & * & * & * & * \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ * \\ * \\ * \end{bmatrix} \begin{bmatrix} 0 & 0 & * & * & * \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ * \\ * \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & * & * \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ * \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & * \end{bmatrix}$$

$$= L_1U_1 + L_2U_2 + L_3U_3 + L_4U_4 + L_5U_5$$

$$= \underbrace{[L_1, L_2, \ldots, L_5]}_{L} \underbrace{\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_5 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} * & & & & \\ * & * & & & \\ * & * & * & & \\ * & * & * & * & \\ * & * & * & * & * \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{bmatrix}}_{U}$$

Solving $Ax = b$ via LU

$l_i u_i^T + l\sim + l\sim \cdots$

$A = LU \in \mathbb{R}^{n\times n}$

L: lower triangular, U: upper triangular

- $\boxed{\text{Cost } \frac{2}{3}n^3 \text{ flops}}$
- For $Ax = b$,
  - first solve $Ly = b$, then $Ux = y$.
  - triangular solve is always backward stable: e.g. $(L + \Delta L)\hat{y} = b$ (see Higham's book)
- Pivoting crucial for numerical stability: $PA = LU$, where P: permutation matrix. Then stability means $\hat{L}\hat{U} = PA + \Delta A$
  - Even with pivoting, unstable examples exist, but still always stable in practice and used everywhere!
- Special case where $A \succ 0$ symmetric positive definite: $A = R^T R$, Cholesky factorization, ALWAYS stable, $\frac{1}{3}n^3$ flops

$Ax = b \Leftrightarrow L\underbrace{Ux}_{y} = b$

$\boxed{Ly = b}$ easy to solve!

1st: $L_{11}y_1 = b_1$, $y_1 = \frac{b_1}{L_{11}}$

2nd: $L_{12}y_1 + L_{22}y_2 = b_2$, $y_2 = \square$

forward subst.

$Ux = y$

upper. $Ux = y$

$U_{nn}x_n = y_n$

x_{n-1}

$\vdots$

x_1

$O(n^2)$

$\lambda_i(A) > 0$

$\Leftrightarrow x^T A x > 0 \quad \forall x \neq 0$

C-F minmax

LU decomposition with pivots

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & & & & \\ A_{31} & & & & \\ A_{41} & & & & \\ A_{51} & & & & \end{bmatrix} = \begin{bmatrix} 1 \\ A_{21}/a \\ A_{31}/a \\ A_{41}/a \\ A_{51}/a \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \end{bmatrix} + \begin{bmatrix} & & & & \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \end{bmatrix}$$

Trouble if $a = A_{11} = 0$! e.g. no LU for $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ solution: pivot, permute rows s.t. largest entry of first (active) column is at top. $\Rightarrow PA = LU$, P: permutation matrix

- $\boxed{PA = LU}$ exists for any nonsingular A (exercise)
- for $Ax = b$, solve $LUx = P^T b$
- cost still $\frac{2}{3}n^3 + O(n^2)$

$PAx = Pb$

$\Rightarrow LUx = Pb$

$PAx = Pb$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} x = b$

P permutation

Cholesky factorisation for $A \succ 0$

SPSD $\lambda_i(A) \geq 0$

If $A \succ 0$ (symmetric positive definite (S)PD$\Leftrightarrow\lambda_i(A) > 0$), two simplifications:

- We can take $U_i = L_i^T =: R_i$ by symmetry $\Rightarrow \frac{1}{3}n^3$ flops
- No pivot needed

$A = LU = R^T R$ not 1's.

suppose $A_2 v = \lambda v$

$\lambda \leq 0$

$v^T A_2 v = \lambda v^T v < 0$

$$A = \underbrace{\begin{bmatrix} * \\ * \\ * \\ * \\ * \end{bmatrix}\begin{bmatrix} * & * & * & * & * \end{bmatrix}}_{R_1 R_1^T} + \underbrace{\begin{bmatrix} & & & & \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \\ & * & * & * & * \end{bmatrix}}_{\text{also PD}}$$

A_2

sym

$A = |- + |- + |-$

$r_1 r_1^T \cdots + r_n r_n^T$

$= R^T R$

consider

$[\alpha \ v^T] A \begin{bmatrix} \alpha \\ v \end{bmatrix} = [\alpha \ v^T] R_1 R_1^T \begin{bmatrix} \alpha \\ v \end{bmatrix} + v^T A_2 v < 0$

$[\alpha\ v^T]\begin{bmatrix} 0 & \\ & A_2 \end{bmatrix}\begin{bmatrix} \alpha \\ v \end{bmatrix}$

Notes:

- diag(R) no longer 1's
- A can be written as $A = R^T R$ for some $R \in \mathbb{R}^{n\times n}$ iff $A \succeq 0$ ($\lambda_i(A) \geq 0$)
- Indefinite case: when $A = A^*$ but A not PSD, $\exists$ $A = LDL^*$ where D diagonal (when $A \in \mathbb{R}^{n\times n}$, D can have 2×2 diagonal blocks), L has 1's on diagonal

choose α s.t. $R_1^T \begin{bmatrix} \alpha \\ v \end{bmatrix} = 0$.

$A = LDL^T$

cont. $x^T A x > 0 \ \forall x \neq 0 \iff A$ PD

$x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$ $x^T A x = A_{11} > 0$.