Courant-Fisher minmax theorem

ith largest eigval λ_i of symmetric/Hermitian A is (below $x \neq 0$)

Rayleigh Quotient $\quad x^TAx \quad \|x\|_2 = 1 \quad$ exercise

$$\lambda_i(A) = \boxed{\max_{\dim \mathcal{S}=i} \boxed{\min_{x\in\mathcal{S}} \boxed{\frac{x^TAx}{x^Tx}}}} \left(= \min_{\dim \mathcal{S}=n-i+1} \max_{x\in\mathcal{S}} \frac{x^TAx}{x^Tx} \right) \qquad (1)$$

$f(S)$

Analogously, for any rectangular $A \in \mathbb{C}^{m\times n} (m \geq n)$, we have

$$\sigma_i(A) = \max_{\dim \mathcal{S}=i} \min_{x\in\mathcal{S}} \frac{\|Ax\|_2}{\|x\|_2} \left(= \min_{\dim \mathcal{S}=n-i+1} \max_{x\in\mathcal{S}} \frac{\|Ax\|_2}{\|x\|_2} \right). \qquad (2)$$

$\min_{x\in S, \|x\|_2=1} \|Ax\|_2$

Proof for (2):

(i) fix S, show $\min_{x\in S, \|x\|_2=1} \|Ax\|_2 \leq \sigma_i(A)$.

(ii) $\sigma_i(A) = \min \|Ax\|$ for some S.

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$\mathrm{span}(S) = [\mathrm{span}[S_1 \cdots S_i]]$

$A = U\Sigma V^T$

Proof for (2):

1. Fix S and let $V_i = [v_i, \ldots, v_n]$. We have trailing
$\dim(\mathcal{S}) + \dim(\mathrm{span}(V_i)) = i + \underbrace{(n-i+1)}_{n-(i-1)} = n+1$, so
$\exists$intersection $w \in S \cap V_i$, $\|w\|_2 = 1$.

$n \quad \underbrace{[S \;\; V_i]}_{n+1} \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = 0 \iff w = Sx_1 = V_i x_2 \neq 0$

can scale s.t. $\|w\|_2 = 1$.

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Proof for (2): $\|Aw\|_2 = \|U\Sigma V^T w\|_2$

b.c. $w = \sum_{k=i}^{n} c_k v_k$, $V_\ell^T w = c_\ell = 0$, $\ell \leq i-1$

1. Fix S and let $V_i = [v_i, \ldots, v_n]$. We have $\dim(\mathcal{S}) + \dim(\text{span}(V_i)) = i + (n - i + 1) = n + 1$, so $\exists$intersection $w \in S \cap V_i$, $\|w\|_2 = 1$.
2. For this w, $\|Aw\|_2 = \|\text{diag}(\sigma_i, \ldots, \sigma_n)(V_i^T w)\|_2 \leq \sigma_i$; thus $\sigma_i(A) \geq \min_{x \in \mathcal{S}} \frac{\|Ax\|_2}{\|x\|_2}$.

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Analogously, for any rectangular $A \in \mathbb{C}^{m \times n} (m \geq n)$, we have

$$\sigma_i(A) \overset{\geq}{=} \max_{\dim \mathcal{S}=i} \underbrace{\boxed{\min_{x \in \mathcal{S}} \frac{\|Ax\|_2}{\|x\|_2}}}_{\sigma_i} \left(= \min_{\dim \mathcal{S}=n-i+1} \max_{x \in \mathcal{S}} \frac{\|Ax\|_2}{\|x\|_2} \right). \qquad (2)$$

$\min\limits_{\|y\|_2=1} \|ASy\| = \sigma_i(A)$

Proof for (2):

1. Fix S and let $V_i = [v_i, \ldots, v_n]$. We have $\dim(\mathcal{S}) + \dim(\text{span}(V_i)) = i + (n - i + 1) = n + 1$, so $\exists$intersection $w \in S \cap V_i$, $\|w\|_2 = 1$.
2. For this w, $\|Aw\|_2 = \|\text{diag}(\sigma_i, \ldots, \sigma_n)(V_i^T w)\|_2 \leq \sigma_i$; thus $\sigma_i(A) \geq \min_{x \in \mathcal{S}} \frac{\|Ax\|_2}{\|x\|_2}$. $\sigma_{\min}(AS) = \sigma_i$
3. For the reverse inequaltiy, take $S = [v_1, \ldots, v_i]$, for which $\boxed{w = v_i}$.

$\|x\|_2=1$, $x \in S$: $Ax = \boxed{AS}y$, $\|y\|=1$; $= U\Sigma_i \hat{y}$, $\text{diag}[\sigma_1 \ldots \sigma_i]$

Weyl's inequality

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Horn & Johnson "Topics"

Analogously, for any rectangular $A \in \mathbb{C}^{m \times n} (m \geq n)$, we have

$$\sigma_i(A) = \max_{\dim \mathcal{S}=i} \min_{x \in \mathcal{S}} \frac{\|Ax\|_2}{\|x\|_2} \quad \left(= \min_{\dim \mathcal{S}=n-i+1} \max_{x \in \mathcal{S}} \frac{\|Ax\|_2}{\|x\|_2} \right).$$

Corollary: Weyl's inequality (Proof: exercise)

- for singular values
  - $\sigma_i(A+E) \in \sigma_i(A) + [-\|E\|_2, \|E\|_2]$ $\forall i = 1,\dots,n$. sig. vals are "well-conditioned"
  - Special case: $\|A\|_2 - \|E\|_2 \leq \|A+E\|_2 \leq \|A\|_2 + \|E\|_2$ $i=1$
- for symmetric eigenvalues

$$\lambda_i(A+E) \in \lambda_i(A) + [-\|E\|_2, \|E\|_2]$$

sym.

Singvals and symmetric eigvals are insensitive to perturbation (well conditioned). Nonsymmetric eigvals are different!

Eigenvalues of nonsymmetric matrices are sensitive

Consider eigenvalues of a Jordan block and its perturbation

$$J = \begin{bmatrix} 1 & 1 & & 0 \\ & 1 & \ddots & \\ 0 & & \ddots & 1 \\ & & & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad J+E = \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ \boxed{\epsilon} & & & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} & 0 \\ \epsilon & \end{bmatrix}$$

$\lambda(J) = \boxed{1}$ (n copies), but $|\lambda(J+E) - 1| \approx \boxed{\epsilon^{1/n}}$

alg. mult. n
geom. mult. 1.

$\leq \|E\|$
$|E|$ if symmetric!

$\epsilon \ll 1$ then

$\epsilon \ll \epsilon^{\frac{1}{n}} \to 1$ as $n \to \infty$

$\|E\| \simeq \epsilon \quad \to \quad |\lambda_i(A) - \lambda_i(A+E)| \simeq \epsilon^{\frac{1}{n}}$

λ_i are very ill-conditioned!!
(sensitive to perturbation)

Matrix decompositions

- SVD $\boxed{A = U\Sigma V^T}$ always
- Eigenvalue decomposition $\boxed{A} = X\Lambda X^{-1}$ usually.
  - Normal: X unitary $X^*X = I$
  - Symmetric: X unitary and Λ real
- Jordan decomposition: $A = XJX^{-1}$, always for square A

$$J = \mathrm{diag}\left(\begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \right)$$

- Schur decomposition $A = \boxed{QTQ^*}$: Q orthogonal, T upper triangular

  eig(A) = diag(T)
- QR: Q orthonormal, $\cancel{U}$ R upper triangular ← $A \in \mathbb{R}^{m\times n}$
- LU: L lower triangular, U upper triangular $A \in \mathbb{R}^{n\times n} \to m\times n$

Red: Orthogonal decompositions, stable computation available