Courant-Fisher minmax theorem

ith largest eigval λ_i of symmetric/Hermitian A is (below $x \neq 0$)

$$\lambda_i(A) = \max_{\dim \mathcal{S}=i} \min_{x \in \mathcal{S}} \frac{x^T A x}{x^T x} \quad \left(= \min_{\dim \mathcal{S}=n-i+1} \max_{x \in \mathcal{S}} \frac{x^T A x}{x^T x} \right) \qquad (1)$$

Analogously, for any rectangular $A \in \mathbb{C}^{m \times n} (m \geq n)$, we have

$$\sigma_i(A) = \max_{\dim \mathcal{S}=i} \min_{x \in \mathcal{S}} \frac{\|Ax\|_2}{\|x\|_2} \quad \left(= \min_{\dim \mathcal{S}=n-i+1} \max_{x \in \mathcal{S}} \frac{\|Ax\|_2}{\|x\|_2} \right). \qquad (2)$$

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Proof for (2):

1. Fix S and let $V_i = [v_i, \dots, v_n]$. We have $\dim(\mathcal{S}) + \dim(\text{span}(V_i)) = i + (n-i+1) = n+1$, so $\exists$intersection $w \in S \cap V_i$, $\|w\|_2 = 1$.

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2. For this w, $\|Aw\|_2 = \|\text{diag}(\sigma_i, \ldots, \sigma_n)(V_i^T w)\|_2 \leq \sigma_i$; thus $\sigma_i(A) \geq \min_{x\in\mathcal{S}} \frac{\|Ax\|_2}{\|x\|_2}$.

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3. For the reverse inequaltiy, take $S = [v_1, \ldots, v_i]$, for which $w = v_i$.

Weyl's inequality

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$$\lambda_i(A) = \max_{\dim \mathcal{S}=i} \min_{x\in\mathcal{S}} \frac{x^TAx}{x^Tx} \quad \left(= \min_{\dim \mathcal{S}=n-i+1} \max_{x\in\mathcal{S}} \frac{x^TAx}{x^Tx} \right)$$

Analogously, for any rectangular $A \in \mathbb{C}^{m\times n}(m \geq n)$, we have

$$\sigma_i(A) = \max_{\dim \mathcal{S}=i} \min_{x\in\mathcal{S}} \frac{\|Ax\|_2}{\|x\|_2} \quad \left(= \min_{\dim \mathcal{S}=n-i+1} \max_{x\in\mathcal{S}} \frac{\|Ax\|_2}{\|x\|_2} \right).$$

Corollary: Weyl's inequality (Proof: exercise)

- for singular values
  - $\sigma_i(A+E) \in \sigma_i(A) + [-\|E\|_2, \|E\|_2]$
  - Special case: $\|A\|_2 - \|E\|_2 \leq \|A+E\|_2 \leq \|A\|_2 + \|E\|_2$
- for symmetric eigenvalues
  $\lambda_i(A+E) \in \lambda_i(A) + [-\|E\|_2, \|E\|_2]$

Singvals and symmetric eigvals are insensitive to perturbation (well conditioned). Nonsymmetric eigvals are different!

Eigenvalues of nonsymmetric matrices are sensitive

Consider eigenvalues of a Jordan block and its perturbation

$$J = \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad J+E = \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ \epsilon & & & 1 \end{bmatrix}$$

$\lambda(J) = 1$ (n copies), but $|\lambda(J+E) - 1| \approx \epsilon^{1/n}$

Matrix decompositions

- SVD $A = U\Sigma V^T$
- Eigenvalue decomposition $A = X\Lambda X^{-1}$
  - Normal: X unitary $X^*X = I$
  - Symmetric: X unitary and Λ real
- Jordan decomposition: $A = XJX^{-1}$,

  $$J = \text{diag}\left(\begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}\right)$$
- Schur decomposition $A = QTQ^*$: Q orthogonal, T upper triangular
- QR: Q orthonormal, U upper triangular
- LU: L lower triangular, U upper triangular

Red: Orthogonal decompositions, stable computation available