

SVD: the most important matrix decomposition

- ▶ **Symmetric eigenvalue decomposition:** $A = V\Lambda V^T$
for symmetric $A \in \mathbb{R}^{n \times n}$, where $V^T V = I_n$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$.
- ▶ **Singular Value Decomposition (SVD):** $A = U\Sigma V^T$
for any $A \in \mathbb{R}^{m \times n}$, $m \geq n$. Here $U^T U = V^T V = I_n$,
 $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$.

SVD proof: Take Gram matrix $A^T A$ and its eigendecomposition $A^T A = V\Lambda V^T$. Λ is nonnegative, and $(AV)^T (AV)$ is diagonal, so $AV = U\Sigma$ for some orthonormal U . Right-multiply V^T .

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Full SVD: $A = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T$ where $U \in \mathbb{R}^{m \times m}$ orthogonal

if $A \in \mathbb{C}^{m \times n}$, $A = U \Sigma V^*$, $U^* U = V^* V = I_n = V V^*$
 $\neq U U^*$ unless $m = n$,

rank, column/row space, etc

$$A = U \Sigma V^T$$

left singular vec. = r

From the SVD one gets

- ▶ rank r of $A \in \mathbb{R}^{m \times n}$: number of nonzero singular values $\sigma_i(A)$ (= # linearly indep. columns, rows)
- ▶ column space (linear subspace spanned by vectors Ax): span of $U = [u_1, \dots, u_r]$
- ▶ row space: row span of v_1^T, \dots, v_r^T
- ▶ null space: v_{r+1}, \dots, v_n

$$A = \begin{bmatrix} U_1 & \dots & U_r & U_{r+1} & \dots & U_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \\ \vdots \\ v_n^T \end{bmatrix}$$

Order

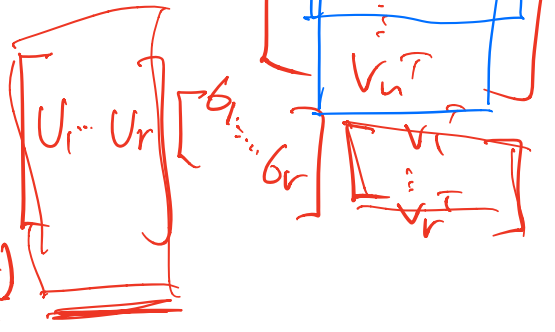
$$A v_{r+1} = 0$$

$$U \Sigma V^T v_{r+1} = U \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} = U \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = 0$$

$\forall i > r$ $\text{span}(\text{row}(A)) = \text{span}(\begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \end{bmatrix})$

$A v_i \neq 0 = \sigma_i u_i$

$\text{span}(\text{col}(A)) = \text{span}(U_1 \dots U_r)$



SVD and eigenvalue decomposition

$\exists v, u$
 $A = U \Sigma V^T$

▶ V eigvecs of $A^T A$

▶ U eigvecs (for nonzero eigvals) of $A A^T$ (up to sign)

▶ $\sigma_i = \sqrt{\lambda_i(A^T A)}$

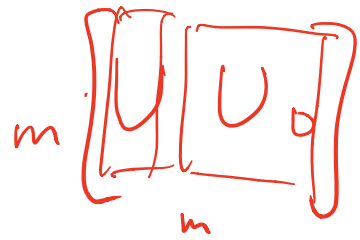
has 0 eigvals if $m > n$

▶ Think of eigenvalues vs. SVD of symmetric matrices, unitary, skew-symmetric, normal matrices, triangular, ...

PSD

▶ Jordan-Wielandt matrix $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$ eigvals $\pm \sigma_i(A)$, and $m - n$ copies of 0. Eigvec matrix is $\begin{bmatrix} U & U_0 \\ V & -V \end{bmatrix}$, $A^T U_0 = 0$

$A = U \Sigma V^T$



orthogonal!
 $U_0^T U = 0_{m-n, n}$

Uniqueness etc

$$A = U \Sigma V^T$$

$$= (U S) \Sigma (S V^T) \leftarrow \text{SVD!}$$

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$$\begin{aligned} S_1^T &= S_1 \\ S_1^2 &= I \end{aligned}$$

extreme case:

$$\begin{aligned} A &= U \text{ orthogonal} \\ &= U \cdot \Sigma \cdot I \\ &= U \cdot \underbrace{Q \cdot Q^T} \\ &= U \cdot \underbrace{I} \cdot \underbrace{V} \end{aligned}$$

only DOF? yes if σ_i are distinct.

No if $\sigma_i = \sigma_{i+1}$

$$\begin{aligned} A &= U \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \end{bmatrix} V^T = U \begin{bmatrix} \sigma_1 & & \\ & \sigma_1 I_2 & \\ & & \ddots \end{bmatrix} V^T = U \begin{bmatrix} \sigma_1 & & \\ & Q & Q^T \\ & & \ddots \end{bmatrix} V^T \\ &= U \begin{bmatrix} I & & \\ & Q & \\ & & I \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \sigma_1 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} I & & \\ & Q & \\ & & I \end{bmatrix} V = \tilde{U} \tilde{\Sigma} \tilde{V}^T \text{ valid SVD} \end{aligned}$$