SVD: the most important matrix decomposition

- **Symmetric eigenvalue decomposition**: $A = V\Lambda V^T$
  for symmetric $A \in \mathbb{R}^{n\times n}$, where $V^TV = I_n$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.
- **Singular Value Decomposition (SVD)**: $A = U\Sigma V^T$
  for any $A \in \mathbb{R}^{m\times n}$, $m \geq n$. Here $U^TU = V^TV = I_n$,
  $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_n$.

maybe 0

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$

$(AB)^T = B^TA^T$

$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

SVD proof:

$\underbrace{A^TA}_{\text{symmetric}} = V\Lambda V^T \succeq 0$

define $B = AV$, $B^TB = V^TA^TAV = \underbrace{V^TV}_{I} \Lambda V^TV = \Lambda \succeq 0$ (diagonal + nonnegative diag.)

$(B^TB)_{ii} = b_i^Tb_i = \|b_i\|_2^2 \geq 0$

ith col of B.

B has orthogonal colums.

if $(B^TB)_{ii} = 0 \Rightarrow b_i = 0$

$B = \begin{bmatrix} u_1 \cdots u_r & 0 \cdots 0 \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \; 0 \end{bmatrix} = \begin{bmatrix} u_1 \cdots u_r \; u_{r+1} \cdots u_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \; 0 \end{bmatrix} = U\Sigma$

left sig vecs

$\sigma_i = \sqrt{\lambda_i}$

So $A = BV^T = \boxed{U}\,\Sigma\,\boxed{V^T}$ ← SV

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SVD proof: Take Gram matrix A^TA and its eigendecomposition $A^TA = V\Lambda V^T$. Λ is nonnegative, and $(AV)^T(AV)$ is diagonal, so $AV = U\Sigma$ for some orthonormal U. Right-multiply V^T.

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Full SVD: $A = U\begin{bmatrix}\Sigma \\ 0\end{bmatrix} V^T$ where $U \in \mathbb{R}^{m\times m}$ orthogonal

if $A \in \mathbb{C}^{m\times n}$, $A = U\Sigma V^*$, $U^*U = V^*V = I_n = VV^*$

$\neq UU^*$ unless $m = n$.

rank, column/row space, etc

From the SVD one gets

- rank r of $A \in \mathbb{R}^{m\times n}$: number of nonzero singular values $\sigma_i(A)$ (=# linearly indep. columns, rows)
- column space (linear subspace spanned by vectors Ax): span of $U = [u_1, \ldots, u_r]$
- row space: row span of $v_1^T, \ldots, v_r^T$
- null space: $v_{r+1}, \ldots, v_n$

$A = U \Sigma V^T$

left sg vec. $\leq r$.

$A = [u_1 \cdots u_r \, u_{r+1} \cdots u_n]\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_n\end{bmatrix}\begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$

Orth.

$A v_{r+1} = 0.$

$U \Sigma V^T v_{r+1}$

$U \begin{bmatrix}\sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{bmatrix}\begin{bmatrix}0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0\end{bmatrix} = U \begin{bmatrix}0 \\ \vdots \\ 0\end{bmatrix} = 0$

$i \leq r$

$A v_i \neq 0$

$= \sigma_i u_i$

span(row(A))

$= \text{span}\begin{pmatrix} v_1^T \\ \vdots \\ v_r^T \end{pmatrix}$

$= [u_1 \cdots u_r]\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{bmatrix}\begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \end{bmatrix}$

span(col(A)) = span($u_1 \cdots u_r$)

SVD and eigenvalue decomposition

$\exists\, V, U$

$A = U \Sigma V^T$

- V eigvecs of $A^T A$
- U eigvecs (for nonzero eigvals) of $A A^T$ (up to sign)

has 0 eigvals if m>n

- $\sigma_i = \sqrt{\lambda_i(A^T A)}$
- Think of eigenvalues vs. SVD of symmetric matrices, unitary, skew-symmetric, normal matrices, triangular,...

PSD

$m+n$

- Jordan-Wieldant matrix $\underbrace{\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}}_{m+n}$: eigvals $\pm\sigma_i(A)$, and $m-n$ copies of 0. Eigvec matrix is $\begin{bmatrix} U & U & U_0 \\ V & -V & 0 \end{bmatrix}$, $A^T U_0 = 0$

$A = U \Sigma V^T$

$m \begin{bmatrix} U & U_0 \end{bmatrix}$, m

orthogonal

$U_0^T U = 0_{m-n,\, n}$

Uniqueness etc

$$A = U\boxed{\Sigma}V^T$$

$$= \underbrace{(U S_i)\Sigma(S_i V^T)}_{} \Leftarrow \text{SVD!}$$

$$\boxed{\begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix}} \qquad S_i^T = S_i \qquad S_i^2 = I$$

extreme case:

$A = U$ orthonormal

$= U \cdot \boxed{I} \cdot \boxed{I}$

$= \underbrace{U Q}_{\hat{U}} \underbrace{A}_{\substack{I \\ \Sigma}} \underbrace{Q^T}_{\hat{V}}$

only DOF? yes if σ_i are distinct.

No if $\boxed{\sigma_i = \sigma_{i+1}}$

$$A = U \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \boxed{\begin{matrix}\sigma_i & \\ & \sigma_{i+1}\end{matrix}} & \\ & & & \sigma_{i+2} \\ & & & & \ddots \end{bmatrix} V^T = U \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \sigma_i I_2 & \\ & & \ddots \end{bmatrix}}_{\Sigma} V^T = U \begin{bmatrix} \sigma_1 & & \\ & \sigma_i Q Q^T & \\ & & \ddots \end{bmatrix} V^T$$

$$= \underbrace{U \begin{bmatrix} I & & \\ & Q & \\ & & I \end{bmatrix}}_{\hat{U}} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_i & \\ & & & \sigma_{i+1} \\ & & & & \ddots \end{bmatrix}}_{} \underbrace{\begin{bmatrix} I & & \\ & \boxed{Q^T} & \\ & & I \end{bmatrix} V}_{\hat{V}} = \hat{U}\Sigma\hat{V}^T \text{ valid SVD}$$

$$I_2 = \boxed{Q}\boxed{Q^T}$$