SVD: the most important matrix decomposition

- **Symmetric eigenvalue decomposition**: $\boxed{A = V\Lambda V^T}$
  for symmetric $A \in \mathbb{R}^{n\times n}$, where $V^TV = I_n$, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$.
- **Singular Value Decomposition (SVD)**: $\boxed{A = U\Sigma V^T}$
  for any $A \in \mathbb{R}^{m\times n}$, $m \geq n$. Here $U^TU = V^TV = I_n$,
  $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_n$.

SVD proof:

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SVD proof: Take Gram matrix A^TA and its eigendecomposition $A^TA = V\Lambda V^T$. Λ is nonnegative, and $(AV)^T(AV)$ is diagonal, so $AV = U\Sigma$ for some orthonormal U. Right-multiply V^T.

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Full SVD: $A = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T$ where $U \in \mathbb{R}^{m\times m}$ orthogonal

rank, column/row space, etc

From the SVD one gets

- rank r of $A \in \mathbb{R}^{m \times n}$: number of nonzero singular values $\sigma_i(A)$ (=# linearly indep. columns, rows)
- column space (linear subspace spanned by vectors Ax): span of $U = [u_1, \ldots, u_r]$
- row space: row span of $v_1^T, \ldots, v_r^T$
- null space: $v_{r+1}, \ldots, v_n$

SVD and eigenvalue decomposition

- V eigvecs of $A^T A$
- U eigvecs (for nonzero eigvals) of AA^T (up to sign)
- $\sigma_i = \sqrt{\lambda_i(A^T A)}$
- Think of eigenvalues vs. SVD of symmetric matrices, unitary, skew-symmetric, normal matrices, triangular,...
- Jordan-Wieldant matrix $\left[\begin{smallmatrix} 0 & A \\ A^T & 0 \end{smallmatrix}\right]$: eigvals $\pm\sigma_i(A)$, and $m-n$ copies of 0. Eigvec matrix is $\left[\begin{smallmatrix} U & U & U_0 \\ V & -V & 0 \end{smallmatrix}\right]$, $A^T U_0 = 0$