Basic linear algebra review

For $A \in \mathbb{R}^{n\times n}$, (or $\mathbb{C}^{n\times n}$; hardly makes difference)

The following are equivalent (how many can you name?):

1. A is nonsingular. $\iff$ A inrevtible, $\exists A^{-1}$. $A^{-1}A = I_n$

$\iff AA^{-1} = I_n$

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The following are equivalent (how many can you name?):

1. A is nonsingular.
2. A is invertible: A^{-1} exists.
3. The map $A : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection.
4. all n eigenvalues of A are nonzero.
5. all n singular values of A are positive.
6. $\mathrm{rank}(A) = n$.
7. the rows of A are linearly independent.
8. the columns of A are linearly independent.
9. $Ax = b$ has a solution for every $b \in \mathbb{C}^n$. $x = A^{-1}b$
10. A has no nonzero null vector. Neither does A^T.
11. A^*A is positive definite (not just semidefinite).
12. $\det(A) \neq 0$.
13. A^{-1} exists such that $A^{-1}A = AA^{-1} = I_n$.
14. ...

Structured matrices

dense + unstructured.

For square matrices,

- Symmetric: $A_{ij} = A_{ji}$ (Hermitian: $A_{ij} = \bar{A}_{ji}$)
  - symmetric positive (semi)definite $A \succ (\succeq) 0$: symmetric and positive eigenvalues

    SPD $\lambda_i > 0$ SPSD $\lambda_i \geq 0$
- Orthogonal: $AA^T = A^TA = I$ (Unitary: $AA^* = A^*A = I$) $\to$ note $A^TA = I$ implies $AA^T = I$

  $(A^*)_{ij} = \bar{A}_{ji}$
- Skew-symmetric: $A_{ij} = -A_{ji}$ (skew-Hermitian: $A_{ij} = -\bar{A}_{ji}$)

  $A = -A^T$ $A = -A^*$
- Normal: $A^TA = AA^T$

  eigvec orthogonal

  $A = VDV^*$

  eig. decomp.
- Tridiagonal: $A_{ij} = 0$ if $|i - j| > 1$
- Triangular: $A_{ij} = 0$ if $i > j$

For (possibly nonsquare) matrices $A \in \mathbb{C}^{m \times n}$, $m \geq n$

$\boxed{A = XDX^{-1}}$ diagonalisable, nonsing.

- Hessenberg: $A_{ij} = 0$ if $i > j + 1$
- "orthonormal": $A^*A = I_n$,

  $AA^* \neq I_m$ if $m > n$
- sparse: most elements are zero

other structures: Hankel, Toeplitz, circulant, symplectic, ...

Vector norms

For vectors $x = [x_1, \ldots, x_n]^T \in \mathbb{C}^n$

- p-norm $\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$
  - Euclidean norm=2-norm $\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$
  - 1-norm $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$
  - ∞-norm $\|x\|_\infty = \max_i |x_i|$

Norm axioms

- $\|\alpha x\| = |\alpha| \|x\|$ for any $\alpha \in \mathbb{C}$
- $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$
- $\|x + y\| \leq \|x\| + \|y\|$) triang. ineq.

Inequalities: For $x \in \mathbb{C}^n$,

- $\frac{1}{\sqrt{n}} \|x\|_2 \leq \|x\|_\infty \leq \|x\|_2$
- $\frac{1}{\sqrt{n}} \|x\|_1 \leq \|x\|_2 \leq \|x\|_1$
- $\frac{1}{n} \|x\|_1 \leq \|x\|_\infty \leq \|x\|_1$

$\| \cdot \|_2$ is **unitarily invariant** as $\|Ux\|_2 = \|x\|_2$ for any unitary U and any $x \in \mathbb{C}^n$.

Matrix norms

For matrices $A \in \mathbb{C}^{m\times n}$,

- p-norm $\|A\|_p = \boxed{\max_x \frac{\|Ax\|_p}{\|x\|_p}}$ $= \max_{\|x\|_p=1} \|Ax\|_p$
  - 2-norm=spectral norm (=operator norm) $\|A\|_2 = \sigma_{\max}(A)$ (largest singular value)
  - 1-norm $\|A\|_1 = \max_i \sum_{j=1}^n |A_{ji}|$
  - ∞-norm $\|A\|_\infty = \max_i \sum_{j=1}^n |A_{ij}|$

  exercise.
- Frobenius norm $\|A\|_F = \sqrt{\sum_i \sum_j |A_{ij}|^2}$ $= \|\mathrm{vec}(A)\|_2$ $= \sqrt{\mathrm{Tr}(A^* A)}$ (2-norm of vectorization)

  $\begin{bmatrix} A_{11} \\ \vdots \\ A_{n1} \\ A_{12} \\ \vdots \\ A_{mn} \end{bmatrix} \in \mathbb{R}^{mn\times 1}$
- trace norm=nuclear norm $\|A\|_* = \sum_{i=1}^{\min(m,n)} \sigma_i(A)$

Red: **unitarily invariant** norms $\|A\| = \|UAV\|$ for any unitary (or orthogonal) U, V

$\mathrm{Tr}(B) = \sum_{i=1}^{n} B_{ii}$

Norm axioms hold for each. Inequalities: For $A \in \mathbb{C}^{m\times n}$, (exercise)

- $\frac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2 \le \sqrt{m}\|A\|_\infty$
- $\frac{1}{\sqrt{m}}\|A\|_1 \le \|A\|_2 \le \sqrt{n}\|A\|_1$
- $\|A\|_2 \le \|A\|_F \le \sqrt{\min(m,n)}\|A\|_2$