Numerical Linear Algebra. QS 3 *(MT 2019)*

Optional questions: Qns 1, 4

1. If $A\,x = b$ and $(A + \delta\,A)\,(x + \delta\,x) = b$ show that

$$\frac{\|\delta\,x\|}{\|x + \delta\,x\|} \le \|A\|\|A^{-1}\|\frac{\|\delta\,A\|}{\|A\|}.$$

2. If Q is orthogonal prove that the condition number for linear systems satisfies $\kappa(Q) = 1$ at least when the $\|\cdot\|_2$ is used. What is the relationship between solving a linear least squares problem via QR factorization and via Cholesky factorization of the normal equations $A^T Ax = A^T b$?

3. Consider $A = \{a_{i,j} : i, j = 1, \ldots, n\}, a_{i,j} = 1/(i + j - 1)$. (See `help hilb` in matlab).

Use `matlab` to compute the condition number for $n = 4, 8, 12$ (`help cond`). For $n = 12$ compute `b=A*ones(n,1)` and then try to recover the solution $x = (1, 1, \ldots, 1)^T$ by Gaussian Elimination which in matlab is the result of `x=A\b`.

4. Explicitly show that if $A \in \mathbb{R}^{n\times n}$ is lower triangular then Gauss–Seidel iteration <u>is</u> forward substitution. This might imply that if A is nearly lower triangle (has few, small entries above the diagonal) then G–S might converge well (fast!). What should you do if you want to apply G–S iteration to a nearly upper triangular matrix?

5. For any $A \in \mathbb{R}^{n\times n}$ show that $\rho(A) \le \|A\|$ in any operator norm.

6. If λ is an eigenvalue of $A \in \mathbb{R}^{n\times n}$ then $(A - \lambda I)\,x = 0$ (*) for some $x \neq 0$. Suppose k is such that $|x_k| = \max_i |x_i|$, then the k^{th} equation of (*) is

$$(a_{kk} - \lambda)\,x_k = -\sum_{j=1, j\neq k}^{n} a_{kj}\,x_j.$$

Deduce that

$$|a_{kk} - \lambda| \leq \sum_{j=1, j\neq k}^{n} |a_{kj}| :$$

you have proved the Gershgorin Circle Theorem: that every eigenvalue of a matrix lies in at least one of the discs $\{z \in \mathbb{C} : \quad |a_{kk} - z| \leq \sum_{j=1,j\neq k}^{n} |a_{kj}|\}$.

For the matrix

$$A = \begin{bmatrix} 9 & 1 & 1 \\ 1 & 5 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

use this theorem to show that the spectral radius of the Jacobi iteration matrix is less or equal to $\frac{2}{3}$.

7. Using matlab on the matrix $A \in \mathbb{R}^{10\times 10}$ which has $a_{ii} = \frac{1}{2}$,

$$a_{ij} = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i < j \end{cases}$$

(try $tril(ones(10, 10)) - \frac{1}{2} * eye(10, 10)$).

Calculate $\|A^k\|_\infty$ for $k = 1, \cdots, 50$ $(norm(A^k), inf)$. What is $\rho(A)$ and how does it relate to what you observe? (see `help for` about loops in matlab).

8. Prove that for the linear system $A\,x = b$, the symmetric SOR method

$$(D + \omega\,L)\;x^{(k+\frac{1}{2})} = \omega\,b + ((1-\omega)\,D - \omega\,U)\;x^{(k)}, \tag{1}$$

$$(D + \omega\,U)\;x^{(k+1)} = \omega\,b + ((1-\omega)\,D - \omega\,L)\;x^{(k+\frac{1}{2})} \tag{2}$$

where $A = D + L + U$, (D is a diagonal matrix, L is a strictly lower triangular matrix, U is a strictly upper triangular matrix), corresponds to the splitting $A = M - N$ where M is the symmetric matrix

$$\frac{1}{\omega\,(2-\omega)}\;(D + \omega\,L)\;D^{-1}\;(D + \omega\,U)\,.$$

9. If A is Strictly Row Diagonally Dominant (SRDD) prove that Jacobi iteration converges for any right hand side b and any stating guess $x^{(0)}$.

10. As in question 9 above, but prove that Gauss-Seidel converges under the same conditions.