MP problem sheet 4 Answers

1/

$$\dot{H} = \delta_H (H_0 - H) + \lambda_H T_S - \mu_H T_p$$

$$c_a \dot{p}_a = \frac{-(p_a - p_v)}{R_c} + H\Delta V$$

$$c_v \dot{p}_v = \frac{p_a - p_v}{R_c} - \frac{p_v}{R_v}$$

$$T_s = g[p_a(t-\tau)] \qquad T_p = 1 - g(p_a)$$

$$g(p) = \frac{1}{1 + \left(\frac{p}{p_0}\right)^n}$$

$$H = H_0 h, \quad p_a = p_0 p, \quad p_v = \frac{R_v p_0}{R_c} q \qquad t \sim \tau$$

$$\Rightarrow g = \frac{1}{1+p^n}$$

$$\Rightarrow \frac{H_0}{\tau} \dot{h} = \delta_H H_0 (1-h) + \lambda_H T_s - \mu_H T_p$$

$$\Rightarrow \frac{1}{\delta_H \tau} \dot{h} = 1 - h + \frac{\lambda_H}{\delta_H H_0} T_s - \frac{\mu_H}{\delta_H H_0} T_p$$

$$\Rightarrow \varepsilon \dot{h} = 1 - h + \lambda g(p_1) - \mu [1 - g(p)]$$

where $\varepsilon = \frac{1}{\delta_H \tau}, \quad \lambda = \frac{\lambda_H}{\delta_H H_0}, \quad \mu = \frac{\mu_H}{\delta_H H_0}$

Also

$$\frac{C_a p_0}{\tau}\dot{p} = -\frac{\left[p_0 p - \frac{R_v p_0}{R_c} q\right]}{R_c} + H_0 \Delta V h$$

$$\frac{C_v}{\tau}\frac{R_v p_0}{R_c}\dot{q} = \frac{p_0 p - \frac{R_v p_0}{R_c} q}{R_c} - \frac{\frac{R_v p_0}{R_c} q}{R_v}$$

$$\Rightarrow \quad \dot{p} = -\frac{\tau}{R_c C_a}\left(p - \frac{R_v}{R_c} q\right) + \frac{H_0 \Delta V \tau}{C_a p_0} h$$

$$\dot{q} = \frac{\tau}{C_v R_v}\left[p - \frac{R_v}{R_c} q - q\right]$$

$$\Rightarrow \quad \dot{p} = \kappa\left[-(p - \delta q) + \sigma h\right]$$

$$\dot{q} = \omega\left[p - (1+\delta) q\right]$$

if $\omega = \frac{\tau}{C_v R_v}$, $\delta = \frac{R_v}{R_c}$, $\kappa = \frac{\tau}{R_c C_a}$, $\sigma = \frac{H_0 \Delta V R_c}{p_0}$

$\delta_H = 1.7\, s^{-1}$

$\tau = 10\, s$

$\lambda_H = 0.84\, s^{-2}$

$\mu_H = 1.17\, s^{-2}$

$H_0 = 1.7\, s^{-1}$

$R_v = 0.06$ mmHg s ml^{-1}

$R_c = 1.2$ " " "

$p_0 = 77$ mmHg

$C_a = 1.5$ ml mmHg^{-1}

$C_v = 50$ ml mmHg^{-1}, $\Delta V = 70$ ml

$\Rightarrow \varepsilon = \frac{1}{\delta_H \tau} \sim 0.06 \quad \lambda = \frac{\lambda_H}{\delta_H H_0} \sim \frac{0.84}{2.89} \sim 0.3$

$\mu = \frac{\mu_H}{\delta_H H_0} \sim \frac{1.17}{2.89} \sim 0.4$, $\omega = \frac{\tau}{C_v R_v} \sim \frac{10}{50 \; 0.06} \sim 3.3$

$\delta = \frac{R_v}{R_c} = \frac{0.06}{1.2} = 0.05$, $\kappa = \frac{\tau}{R_c C_a} = \frac{10}{1.8} \sim 5.5$

$\sigma = \frac{H_0 \Delta V R_c}{p_0} \sim \frac{1.7 \; 70 \; 1.2}{77} \sim 1.85$

Summary

$$\varepsilon \dot{h} = 1 - h + \lambda g(p_1) - \mu[1 - g(p)]$$

(ε: 0.06, λ: 0.3, μ: 0.4)

$$\dot{p} = \kappa[-(p - \delta q) + \sigma h]$$

(κ: 5.5, δ: 0.05, σ: 1.85)

$$\dot{q} = \omega[p - (1+\delta)q], \quad g = \frac{1}{1+p^n}$$

(ω: 3.3, δ: 0.05)

$\delta \ll 1$ so we lose the δq term in the p equation

So $\delta \downarrow 0$. Steady state $p = \sigma h$

$$1 - h + \lambda g - \mu + \mu g = 0$$

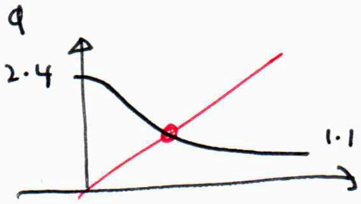
$$\text{so} \quad h = 1 - \mu + (\lambda + \mu) g$$

$$\text{so} \quad p = \sigma(1-\mu) + \sigma(\lambda+\mu) g$$

$$\equiv \phi(p) = A + \frac{B}{1+p^n}$$

$$A = \sigma(1-\mu) \simeq 1.85 \times 0.6 \simeq 1.1$$

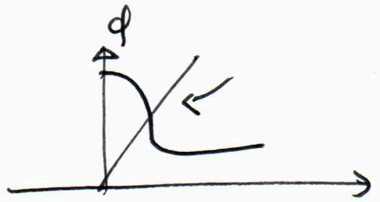
$$B = \sigma(\lambda+\mu) \simeq 1.85 \times 0.7 \simeq 1.3$$



$\rightsquigarrow$ Unique positive solution

Which you iterate depends on $|\varphi'(p^*)|$

I guess $n=7$



$|\varphi'|>1$

so try $p_{r+1} = \varphi^{-1}(p_r)$ i.e.

$$p_r = A + \frac{B}{1+p_{r+1}^n}$$

so $$1+p_{r+1}^n = \frac{B}{p_r - A}$$

$$p_{r+1} = \left[\frac{B}{p_r-A} - 1\right]^{\frac{1}{n}}$$ (note start with $p_0 > A$)

on a calculator, $n=7$, $A = 1.1$, $B = 1.3$

$p_0 = 1.5$
$p_1 = 1.1228\ldots$
$p_2 = 1.777\ldots$ oops
$p_3 =$

try $p_{r+1} = A + \frac{B}{1+p_r^n}$, $n=7$

$p_0 = 1.5$
$p_1 = 1.1718\ldots$
$p_2 = 1.422\ldots$ better
$p_3 = 1.2018\ldots$

converging slowly.

just plot $p - A - \frac{B}{1+p^n}$ → $p_\infty \approx 1.29$

Now if we expect p^* (steady state) = 100 mmHg

$\Rightarrow p$ (non-d) $= \frac{100}{77} = 1.2987\ldots$

So then

$$p = A + \frac{B}{1+p^n}$$

$$\Rightarrow \quad 1+p^n = \frac{B}{p-A}$$

$$n \ln p = \ln\left[\frac{B}{p-A} - 1\right]$$

So $n = \dfrac{\ln\left[\frac{B}{p-A} - 1\right]}{\ln p}$ use $A = 1.1$, $B \sim 1.3$, ~~1.2987~~ $p = 1.2987$

$\Rightarrow n \simeq 6.552$

2

(i) Consider $f(\sigma) = \frac{e^{-\sigma}}{\sigma+\beta} = -\frac{1}{\gamma}$

f has an isolated essential singularity at ∞

$f \neq 0$ anywhere

$\therefore$ Picard $\Rightarrow$ $f = -\frac{1}{\gamma}$ in any neighbourhood of ∞

(ii) If $\sigma \to \infty$

then $-\gamma e^{-\sigma} \to \infty \Rightarrow \mathrm{Re}\,\sigma \to -\infty$

(iii) $\sigma = -\beta - \gamma e^{-\sigma}$

$\frac{d\sigma}{d\gamma} = \sigma' = -e^{-\sigma} + \gamma e^{-\sigma}\sigma'$

$\Rightarrow \sigma' = \frac{-e^{-\sigma}}{1-\gamma e^{-\sigma}} = \frac{-\gamma e^{-\sigma}}{\gamma[1-\gamma e^{-\sigma}]} = \frac{\sigma+\beta}{\gamma[1+\beta+\sigma]}$

or: $\gamma(\sigma)$ is analytic $\Rightarrow \sigma(\gamma)$ is also except where $\gamma' = 0$

(iv) $\gamma \ll 1$. [If $\sigma \lesssim O(1)$ $\sigma \simeq -\beta$]

~~Alternatively~~ Suppose $\mathrm{Re}\,\sigma > 0$

then $\sigma + \beta = -\gamma e^{-\sigma}$

$|\sigma+\beta| > \mathrm{Re}\,\sigma + \beta > \beta$

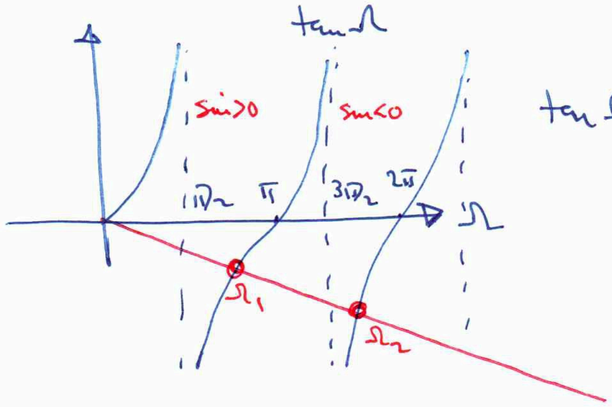
So $\beta < \gamma |e^{-\sigma}| = \gamma e^{-\mathrm{Re}\,\sigma} < \gamma$

$\Rightarrow \mathrm{Re}\,\sigma < 0$ for $\gamma < \beta$.

(v) Hence instability occurs if $\sigma = i\Omega$ (not $\sigma = 0$ as never a root)

$\Rightarrow i\Omega = -\beta - \gamma(\cos\Omega - i\sin\Omega)$

$\Rightarrow \gamma = \frac{\Omega}{\sin\Omega}$, $-\beta = \gamma\cos\Omega = \frac{\Omega}{\tan\Omega}$



$\tan\Omega = -\frac{\Omega}{\beta}$

∞ roots $\Omega_1 \in (\pi/2, \pi)$

$\Omega_2 \in (3\pi/2, 2\pi)$

$\Omega_3 \in (5\pi/2, 3\pi)$ etc.

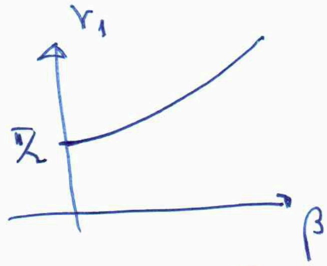
of which $\gamma_k > 0$ for $k = 1, 3, \dots$

[For large k odd $\Omega_k \simeq (k-\frac{1}{2})\pi$

$\gamma_k \simeq (k-\frac{1}{2})\pi$

$\gamma_{2n+1} \simeq (2n+\frac{1}{2})\pi \quad n = 0, 1 \dots$]

Instability first at γ_1



[see fig 6.11 of notes]

To check transversality:

$$\sigma' = \frac{\beta + i\Omega}{\gamma(1+\beta+i\Omega)}$$

$$= \frac{(\beta+i\Omega)(1+\beta-i\Omega)}{\gamma[(1+\beta)^2+\Omega^2]}$$

$$= \frac{\beta(1+\beta)+\Omega^2+i\Omega}{\gamma[(1+\beta)^2+\Omega^2]}$$

$\mathrm{Re}\,\sigma' > 0$ for $\beta, \gamma > 0$

So roots cross successively & remain unstable.

[Note $\tan\Omega = -\frac{\Omega}{\beta}$

$$\Rightarrow 1 + \frac{\Omega^2}{\beta^2} = 1 + \tan^2\Omega = \sec^2\Omega$$

$$\cos^2\Omega = \frac{\beta^2}{\beta^2+\Omega^2}$$

$$\sin^2\Omega = \frac{\Omega^2}{\beta^2+\Omega^2}$$

$$\sin\Omega = \frac{\Omega}{(\beta^2+\Omega^2)^{1/2}}$$

$\gamma = (\beta^2+\Omega^2)^{1/2}$ increases with Ω]

[Note:

$\gamma = \frac{\Omega}{\sin\Omega}$, $\frac{\Omega}{\tan\Omega} = -\beta \Rightarrow -\frac{\beta}{\gamma} = \cos\Omega$

$$\gamma'(\beta) = \frac{\gamma'(\Omega)}{\beta'(\Omega)} = \frac{\frac{1}{\sin\Omega} - \frac{\Omega\cos\Omega}{\sin^2\Omega}}{-\left[\frac{1}{\tan\Omega} - \frac{\Omega\sec^2\Omega}{\tan^2\Omega}\right]}$$

$$= \frac{\sin\Omega - \Omega\cos\Omega}{-[\sin\Omega\cos\Omega - \Omega]} = \frac{1+\beta}{-[\cos\Omega - \frac{\Omega}{\sin\gamma}]}$$

$$\Rightarrow \gamma' = \frac{\gamma(1+\beta)}{\beta+\gamma^2} > 0$$

from graph at top, $\beta \to 0$, $\Omega_1 \to \pi/2$, $\gamma_1 \to \pi/2$

$\beta \to \infty$, $\Omega_1 \to \pi$, $\Omega_1 = \pi - \delta$, $\frac{\pi}{\delta} \approx \beta$

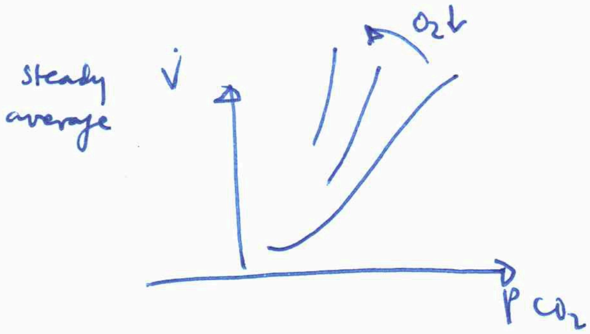
$\gamma_1 = \frac{\pi}{\delta} \Rightarrow \gamma_1 \sim \beta$ as $\beta \to \infty$]

3. Minute ventilation:

volume of air breathed in a minute

Central chemoreceptor responds to H^+ & thus effectively to CO_2 via the bicarbonate buffering system.

Peripheral chemoreceptors also respond to CO_2 but the response is amplified if O_2 is lowered



Mackey–Glass

$$K\dot{p} = M - p\dot{V}$$

$$\dot{V} = \dot{V}(p_\tau)$$

K tissue volume

M metabolic production rate

$\dot{V}$ ventilation

1st eq conservation of CO_2

2nd controller

$$\dot{V} = G[p - p_0]_+$$

$M = 200$ mm Hg ℓ (BTPS) min^{-1}

$p_0 = 35$ mm Hg

$K = 40\ \ell$ (BTPS)

$G = 2\ \ell$ (BTPS) min^{-1} mm Hg^{-1}

$\tau = 0.2$ min

$$K\dot{p} = M - p\underbrace{G(p_\tau - p_0)_+}_{\dot{V}}$$

Define $p = p_0 + |\Delta p| p^*$
$\dot{V} = G \Delta p\, v$ $\Rightarrow v = [p^*_1]_+$
$t \sim \tau$

and (drop $*$) $\frac{K\Delta p}{\tau}\dot{p} = M - p_0\left(1 + \frac{\Delta p}{p_0} p\right) G \Delta p\, v$

Choose $M = p_0 G \Delta p$ i.e $\Delta p = \frac{M}{p_0 G} \sim \frac{200 \text{ mm Hg } \ell \text{ min}^{-1}}{35 \text{ mm Hg } 2\ \ell \text{ min}^{-1} \text{ mm Hg}^{-1}}$
≈ 3 mm Hg

$\mu = \frac{\Delta p}{p_0} = \frac{M}{p_0^2 G}$

then $\dot{p} = \frac{M\tau}{K\Delta p}\left[1 - (1 + \mu p) v\right]$

$= \alpha[1 - (1 + \mu p) v]$

$\alpha = \frac{M\tau}{K\Delta p} = \frac{G p_0 \tau}{K}$

So $\alpha \sim \frac{2\ \ell \text{ min}^{-1} \text{ mm Hg}^{-1}\ 35 \text{ mm Hg}\ 0.2 \text{ min}}{40\ \ell}$

$\simeq \frac{1}{3}$ & $\mu \simeq \frac{3}{35} \simeq 0.08$

Conversion factor from wet saturated body temperature to dry NTP : $863 = 760 \times \frac{310}{273}$

4/

$$K_L \dot{P}_{aCO_2} = -\dot{V} P_{aCO_2} + 863\, K_{CO_2} Q \left[P_{vCO_2} - P_{aCO_2} \right] \quad (1)$$

(K_L: lung volume; $\dot{V} P_{aCO_2}$: ventilatory loss; K_{CO_2}: solubility coefficient; Q: blood flow; P_{vCO_2}: venous; P_{aCO_2}: arterial)

$$K_{CO_2} K_B \dot{P}_{BCO_2} = MR_{TBCO_2} + K_{CO_2} Q_B \left[P_{aCO_2}(t - \tau_{aB}) - P_{BCO_2} \right] \quad (2)$$

(K_B: brain volume; MR_{TBCO_2}: production rate in [redacted] brain; Q_B: brain blood flow; τ_{aB}: arterial delay to brain)

$$K_{CO_2} K_T \dot{P}_{TCO_2} = MR_{TCO_2} + (Q - Q_B) K_{CO_2} \left[P_{aCO_2}(t - \tau_{aT}) - P_{TCO_2} \right] \quad (3)$$

(K_T: tissue volume; MR_{TCO_2}: tissue production rate; τ_{aT}: delay to tissues)

$$Q P_{vCO_2} = Q_B P_{BCO_2}(t - \tau_{vB}) + (Q - Q_B) P_{aCO_2}(t - \tau_{aT}) \quad (4)$$

(τ_{aT}: delay from tissues)

(1) : conservation of arterial/lung CO_2

(2) : " brain CO_2

(3) : " tissue CO_2

(4) : venous blood flow CO_2 is the sum of the flow from the brain & that from other tissues

Response time scales (green terms)

(1) $t_a = \dfrac{K_L}{863 K_{CO_2} Q}$ (since $863 K_{CO_2} Q \gg \dot{V}$) $\sim \dfrac{3}{26} \simeq 0.12$ min

(2) $t_B = \dfrac{K_B}{Q_B} \sim \dfrac{1}{0.75} = \dfrac{4}{3}$ min $= 80$ s

(3) $t_T = \dfrac{K_T}{Q} \sim \dfrac{39}{6} \sim 6.5$ min

Therefore on a time scale of a minute P_{aCO_2} is rapid →

P_{aCO_2} → quasi-equilibrium.

t_T is slow so P_{TCO_2} changes slowly

$$(*) \Rightarrow \quad P_{vCO_2} = P_{TCO_2} + \frac{Q_B}{Q}\left[P_{BCO_2} - P_{TCO_2}\right]$$

and since $\frac{Q_B}{Q} \simeq 0.15$,

$$P_{vCO_2} \approx P_{TCO_2} \text{ is slowly varying}$$

but also

$$\frac{-\dot{V}}{863 K_{CO_2} Q} P_{aCO_2} + P_{vCO_2} - P_{aCO_2} \approx 0$$

$$\Rightarrow \quad P_{aCO_2} \approx \frac{P_{vCO_2}}{1 + \frac{\dot{V}}{863 K_{CO_2} Q}} \approx \frac{P_{TCO_2}}{1 + \frac{\dot{V}}{863 K_{CO_2} Q}}$$

gives P_{aCO_2}

[redacted]

[redacted]

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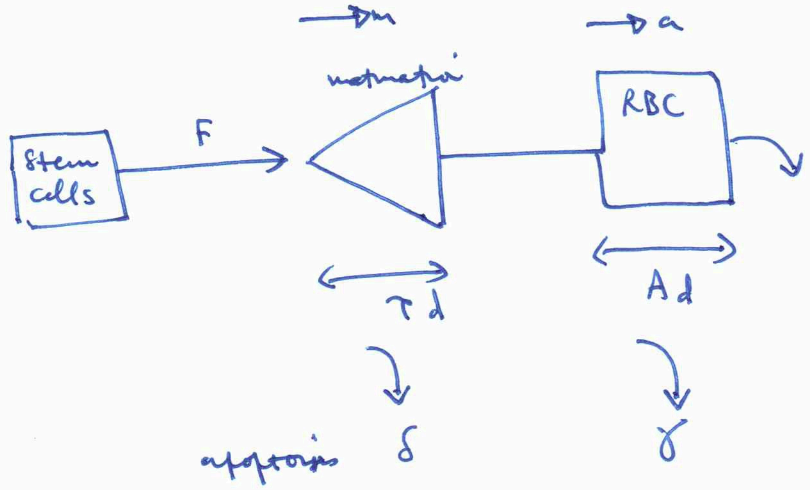


Let $p(t,m)$ be the proliferative cell density (m is maturation time)

$$\Rightarrow \frac{\partial p}{\partial t} + \frac{\partial p}{\partial m} = -\delta p, \quad 0<m<\tau$$

Let $e(t,a)$ be circulating RBC density as function of age a

$$\Rightarrow \frac{\partial e}{\partial t} + \frac{\partial e}{\partial a} = -\gamma e, \quad 0<a<A$$

Now with $P = \int_0^\tau p \, dm$

we have $\frac{dP}{dt} + p|_{m=\tau} - p|_{m=0} = -\delta P$

ie $\frac{dP}{dt} = p|_{m=0} - \delta P - p|_{m=\tau}$

clearly $p|_{m=0}$ = supply rate $= F$

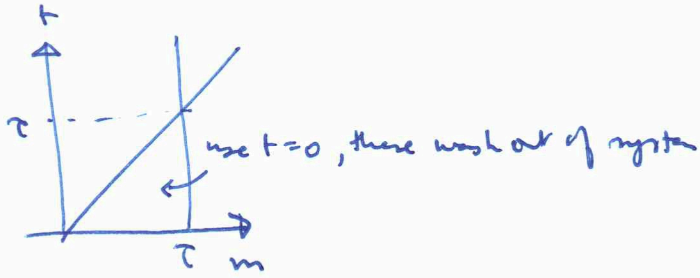
& $p|_{m=\tau}$ = loss rate to circulation

(thus $e|_{a=0} = p|_{m=\tau}$)

Characteristics :

$\dot{p} = -\delta p$

$\dot{m} = 1$



Use boundary conditions on m=0

$m=0$

$t=s$

$p = F(s)$ (specifically $F[E(s)]$, $E(t) = \int_0^A e(t,a)\,da$

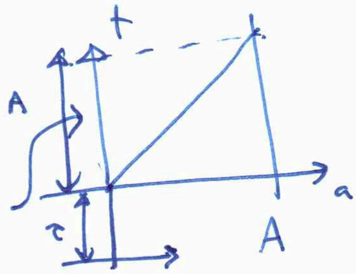
$\Rightarrow p = F(s)\, e^{-\delta(t-s)}$

$m = t-s$

So $p = F(t-m)\,e^{-\delta m}$ $\forall m$ if $t > \tau$

$\Rightarrow p|_{m=\tau} = e|_{a=0} = F(t-\tau)e^{-\delta\tau}$

Next



$\dot{e} = -\gamma e$

$\dot{a} = 1$

$$\left\{ \begin{array}{l} a=0 \\ t=\xi \\ e = F(\xi-\tau)e^{-\delta\tau} \end{array} \right.$$

$e = F(\xi-\tau)e^{-\delta\tau} e^{-\gamma(t-\xi)}$ ← can only use this for $\xi > \tau$

$a = t-\xi$ viz $t > a+\tau$

$\Rightarrow e = F(t-a-\tau)e^{-\delta\tau-\gamma a}$

and works for all a if $t > A+\tau$

Thus with $E = \int_0^A e\,da$

$$\dot{E} = -\gamma E + e|_{a=0} - e|_{a=A}$$

$$= -\gamma E + F[E(t-\tau)]e^{-\delta\tau} - F[E(t-A-\tau)]e^{-\delta\tau-\gamma A}$$

If there is no age limit, then $A \to \infty$

& just $\dot{E} = -\gamma E + f(E_\tau) e^{-\delta\tau}$

Such reduction occurs if $\gamma A \gg 1$ for example

Non-d $\quad f = f_0 g$

& scale $E \sim E_0 = \frac{f_0 e^{-\delta\tau}}{\gamma}$, $t \sim \tau$

then non-d $\frac{E_0}{\tau}\dot{E} = -\gamma E_0 E + f_0 g(E_1) e^{-\delta\tau} - f_0 g\left[E\left(t-1-\frac{A}{\tau}\right)\right] e^{-\delta\tau - \gamma A}$

thus $\frac{1}{\gamma\tau}\dot{E} = -E + g(E_1) - e^{-\gamma A} g\left[E\left(t-1-\frac{A}{\tau}\right)\right]$

define $\mu = \gamma\tau$, $\Lambda = \frac{A}{\tau}$ $\Rightarrow \gamma A = \mu\Lambda$

& $\dot{E} = \mu\left[-E + g(E_1) - e^{-\mu\Lambda} g(E_{1+\Lambda})\right]$

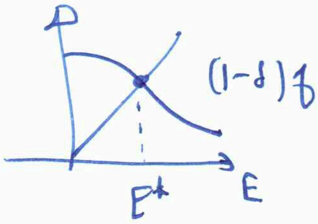
$A = 120$ d $\tau = 6$ d, $\Rightarrow \Lambda = 20$

If μ is not small then the finite lifetime is irrelevant – so why would A be so big. Also we would hope E would be relatively constant & this is facilitated by $\mu \ll 1$.

Write $\delta = e^{-\mu\Lambda} < 1$

Steady state $E = (1-\delta)\, f(E)$

f is decreasing $\Rightarrow$ unique steady state



Linearise $E = E^* + \eta$

$$\Rightarrow \dot{\eta} \simeq \mu\left[-\eta + f'\eta_1 - \delta f'\eta_{1+\Lambda}\right]$$

$\eta = e^{\sigma t}$

$$\Rightarrow \sigma = \mu\left[-1 - |f'|\left\{e^{-\sigma} - \delta e^{-\sigma(1+\Lambda)}\right\}\right]$$

$$\Rightarrow \sigma = -\mu\left[1 + |f'|e^{-\sigma}\left\{1 - \delta e^{-\sigma\Lambda}\right\}\right]$$

Suppose that $\mathrm{Re}\,\sigma > 0$:

this requires $|f'|\left|e^{-\sigma}\left\{1-\delta e^{-\sigma\Lambda}\right\}\right| > 1$

but LHS $< |f'|\,|1-\delta e^{-\sigma\Lambda}|$

& since $\delta < 1$, $|e^{-\sigma\Lambda}| < 1$, certainly $|1-\delta e^{-\sigma\Lambda}| < 2$

So LHS $< 2|f'|$

So E^* is stable if $|f'| < \frac{1}{2}$

Since $\Lambda \gg 1$, $|1-\delta e^{-\sigma\Lambda}| \simeq 1$ & this becomes $|f'| < 1$

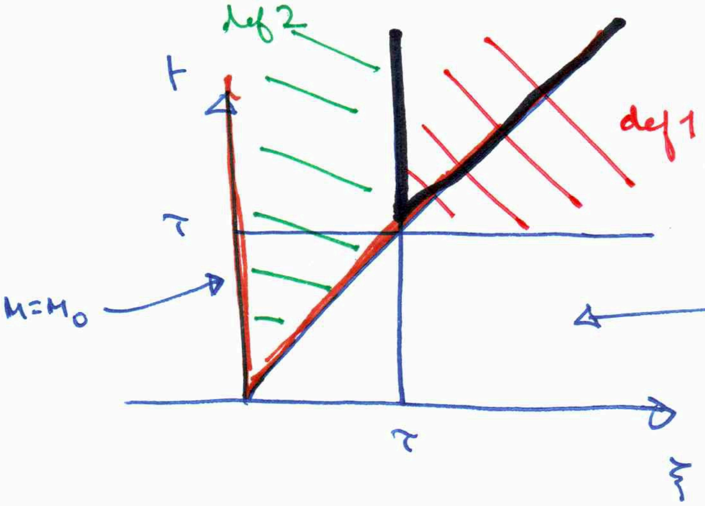
[unless $\mathrm{Re}\,\sigma \approx 0$ ....]

6. $M_t + M_\xi = -RM + Q$

$$Q = 2e^{-\gamma\tau} R(t-\tau, \xi-\tau) M(t-\tau, \xi-\tau) \qquad \xi > \tau, \ t > \tau \qquad (1)$$

$$= 2e^{-(\gamma_0+v_0)\tau} e^{(\gamma_0+v_0-\gamma)\xi} v_0 R_0(t-\tau) N_0(t-\tau), \qquad (2)$$

$$\xi < \tau, \ t > \xi$$



The definition of Q will depend on the initial (t=0) condition but is washed out of the system.

We focus on the region $t > \xi$.

Using $R = (1+\lambda)R_0$, $\lambda = 2e^{-\gamma\tau} - 1$, $\gamma_0 + v_0 = \gamma$, [$N_0 v_0 = M_0$] we have

$$Q = \begin{cases} (1+\lambda) R M_{\tau,\tau} & (1) \\ R M_0 & (2) \end{cases}$$

and $M = M_0$ on $\xi = 0$, $t > 0$.

In $t > \xi$, $\xi < \tau$ (def 2) we have

$$M_t + M_\xi = -RM + RM_0, \quad M = M_0 \text{ on } \xi = 0$$

& the solution is obviously $M \equiv M_0$

Therefore to solve the model in $t > \xi$, $t > \tau$ (outlined in black)

we have the characteristic equations

$$\dot{M} = -RM + (1+\lambda) R M_{\tau,\tau}$$
$$\dot{\xi} = 1$$

& $\begin{cases} M = M_0 \\ t = s > \tau \\ \xi = \tau \end{cases}$

Thus $\xi = t - s + \tau$.

Since the initial condition is independent of t, it is clear that the solution is a function of ξ only

Defining $\eta = \dfrac{\xi - \tau}{\tau}$, $\overset{M(t,\xi)}{\overset{\parallel}{M}} = M_0 u(\eta)$

we have $\eta = \dfrac{t-s}{\tau}$ and thus $\dot{M} = \dfrac{M_0}{\tau} u' = -RM_0 u + (1+\lambda) R u(\eta - 1)$,

since $M_{\tau,\tau} = M(t-\tau, \xi - \tau) = M_0 u\left[\dfrac{\xi - \tau - \tau}{\tau}\right] = M_0 u(\eta - 1)$

i.e.

$$\boxed{u' = -\alpha u - \Gamma u_1}$$

where $\alpha = R\tau$, $\Gamma = -(1+\lambda)R\tau$

Note that applies for $\eta \geq 0$ $(\xi > \tau)$, and the initial function is $u = 1$ for $\eta \in [-1, 0)$

Next, define the Laplace transform

$$U(p) = \int_0^\infty u(\eta) e^{-p\eta}\, d\eta$$

Note

$$\hat{u}_1 = \int_0^\infty u(\eta - 1) e^{-p\eta}\, d\eta$$

$[\eta = 1 + s]$

$$= \int_{-1}^\infty u(s) e^{-p} e^{-ps}\, ds$$

$$= e^{-p} U + e^{-p} \int_{-1}^0 e^{-ps}\, ds$$

$$= e^{-p} U + \frac{e^{-p}}{p}\left[e^p - 1\right] = e^{-p} U + \frac{(1 - e^{-p})}{p}$$

So we have from $u' = -\alpha u - \Gamma u_1$, $u=1$, $\eta<0$

$$pU - 1 = -\alpha U - \Gamma\left[e^{-p}U + \frac{(1-e^{-p})}{p}\right]$$

or $$U = \frac{1 - \Gamma\left(\frac{1-e^{-p}}{p}\right)}{p + \alpha + \Gamma e^{-p}}$$

Hence $$U = \frac{p - \Gamma + \Gamma e^{-p}}{p(p + \alpha + \Gamma e^{-p})} = \frac{p + \alpha + \Gamma e^{-p} - (\alpha+\Gamma)}{p(p+\alpha+\Gamma e^{-p})}$$

$$= \frac{1}{p} - \frac{\alpha+\Gamma}{p(p+\alpha+\Gamma e^{-p})}$$

$$= \frac{1}{p} + \frac{(-\alpha-\Gamma)e^{p}}{p[(p+\alpha)e^{p}+\Gamma]} = \frac{1}{p} + \frac{\Lambda e^{p}}{p[(p+\alpha)e^{p}+\Gamma]}$$

where $\Lambda = -\alpha - \Gamma = -R\tau + (1+\lambda)R\tau = \lambda R\tau$

To invert this, we have $$u = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} U(p)\, e^{p\eta}\, dp$$

with all singularities of U in $\operatorname{Re} p < \gamma$.

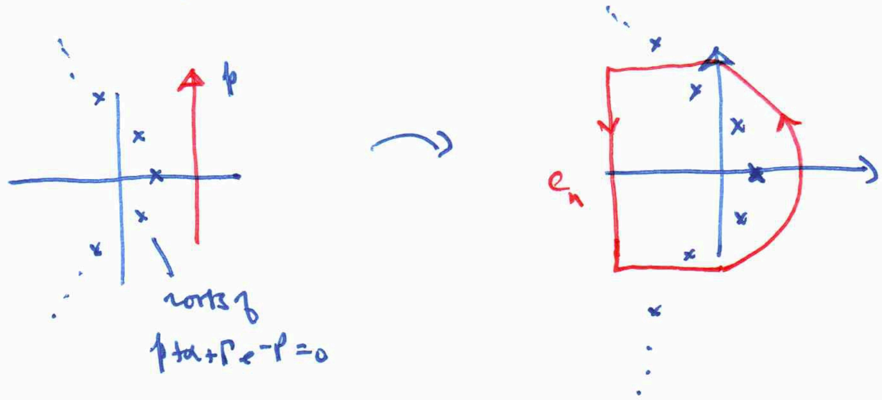
The singularities of U are at $p = 0$ (removeable, so does not contribute)

& where $p = -\alpha - \Gamma e^{-p}$

– but we know all about this! (see question 5)

For $-\Gamma > \alpha$ as here, there is one +ve root & ∞ complex roots, with a finite number to the right of the imaginary axis.

So the inversion contour can be taken as the limit of C_n as $n \to \infty$, where C_n is as shown



where the left hand part is the left side of a square of half-length $(n+\frac{1}{2})\pi$. The integral round this part vanishes as $n \to \infty$ by a version of Jordan's lemma, provided $p + \alpha + \Gamma e^{-p}$ is bounded away from zero on top + bottom (clearly it goes to ∞ on the left side).

But for $p = r + i(n+\frac{1}{2})\pi$, $p + \alpha + \Gamma e^{-p} = r + \alpha + i[(n+\frac{1}{2})\pi + |\Gamma| e^{-r}]$ if n is even & $\Gamma < 0$, so $|p + \alpha + \Gamma e^{-p}| > n\pi \to \infty$ in fact.

Therefore

$$u(\eta) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{C_n} U(p)\, e^{p\eta}\, dp$$

$$= \sum_j c_j\, e^{p_j \eta} ,$$

where c_j are the residues of U at the roots

of $p + \alpha + \Gamma e^{-p} = 0$

Since $U = \dfrac{\Lambda}{p\left[p+\alpha+\Gamma e^{-p}\right]} + \dfrac{1}{p}$

& $(p+\alpha+\Gamma e^{-p})' = 1 - \Gamma e^{-p}$, these are

$$\frac{1}{p_j} \cdot \frac{\Lambda}{1-\Gamma e^{-p_j}}$$

[redacted] $= \dfrac{1}{p_j}\, \dfrac{\Lambda}{(1+p_j+\alpha)}$

[Note : the roots $p+\alpha+\Gamma e^{-p}=0$

satisfy $p_n = \ln\left(\dfrac{p_n+\alpha}{\Gamma}\right) = \ln p_n - \ln\Gamma + \ln\left(1+\dfrac{\alpha}{p_n}\right) + 2n i\pi$

or $p_n = 2n i\pi\left[1 + \dfrac{\left\{\ln p_n - \ln\Gamma + \frac{\alpha}{p_n} \cdots\right\}}{2n i\pi}\right]$

& $\ln p_n = \ln 2n i\pi + \dfrac{\ln p_n}{2n i\pi} \cdots$

$= (\ln 2n i\pi)(1 + O(\tfrac{1}{n}))$

thus $p_n \simeq 2n i\pi\left[1 + \dfrac{\ln 2n i\pi}{2n i\pi} \cdots\right] = 2n i\pi + \ln 2n i\pi + \cdots$

& $p_j \sim O(j)$ as $j\to\infty$.]