MP lecture notes q 3·4.

Answer follows

There are two fairly straightforward misprints

1 : w_M is the unique ~~+~~ +ve number s.t. $f(v) = w_M$ has a double root

2 : show $\frac{\partial u'}{\partial u} + \frac{\partial v'}{\partial v} > 0$

But

To use the comparison method to show uniqueness of w_c requires

$\frac{\partial g(V, w_c)}{\partial w_c} < 0$ unfortunately – but this is not true – only for

$V < v_- - v_c$ (see page 3)

That there is ~~one~~ value of w_c to connect ~~~~ C to D

is straightforward (p 6) but uniqueness needs another

comparison method – which is actually easier than the question as

stated! (p7.).

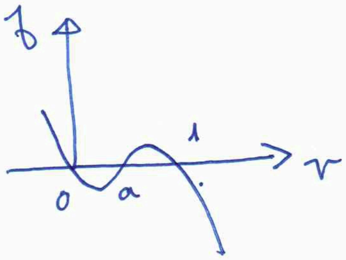
Math Physiol lecture notes

q3.4 answer

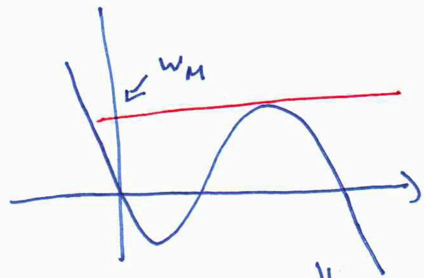
$$v' = u$$

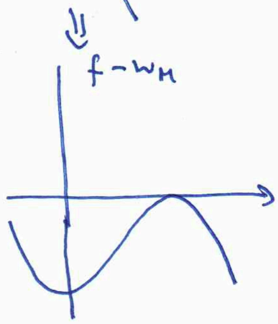
$$u' = cu - [f(v) - w_c]$$

$f = v(1-v)(v-a)$ $\quad 0<a<1$



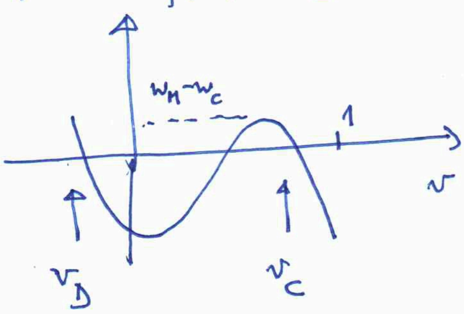
so





so if $0 < w_c < w_M$

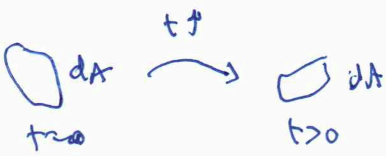
then



$$\frac{\partial u'}{\partial u} + \frac{\partial v'}{\partial v} = c > 0$$

implies that areas increase with ξ

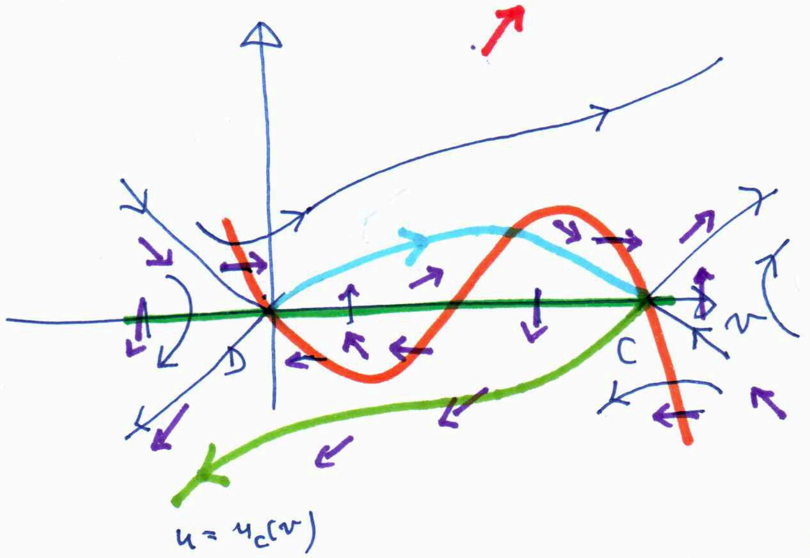
$$\left(\frac{d}{d\xi} dA = J\, dA \;, \; J = \frac{\partial(u', v')}{\partial(u, v)} \right)$$



2

$w_c = 0$

$v' = u$
$u' = cu - f(v)$



nullclines + v
− u

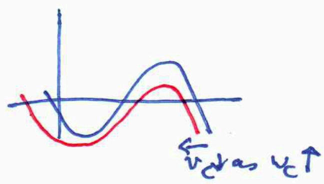
direction at $u \to \infty$
$v' > 0$
$u' > 0$ $(c > 0)$

fill in other trajectories (+ given DC separatrix)

The separatrix in $u < 0$ must be as shown: the alternative is that it reaches $u = 0$ at E where $0 < v_E < v_C = 1$ ($v_C = v_D$) then the area enclosed by the separatrix and ED would decrease (or remain constant if E = D) ✗. Hence $u_c(v_D) < 0$ when $w_c = 0$

Next $v = v_c - V$, $g(V, w_c) = f(v) - w_c = f(v_c - V) - w_c$

so $\dfrac{\partial g}{\partial w_c} = f'(v)\dfrac{\partial v_c}{\partial w_c} - 1$ (clearly v_c decreases with w_c



In fact $f(v_c) \equiv w_c$

so $f'(v_c)\dfrac{\partial v_c}{\partial w_c} = 1$

so $\dfrac{\partial g}{\partial w_c} = \dfrac{f'(v)}{f'(v_c)} - 1 = \dfrac{f'(v_c) - f'(v)}{-f'(v_c)}$; $f'(v_c) < 0$

so $\dfrac{\partial g}{\partial w_c} < 0$ if $f'(v) > f'(v_c)$

$$\frac{\partial g}{\partial w_c} < 0 \quad \text{for } f'(v) > f'(v_c)$$

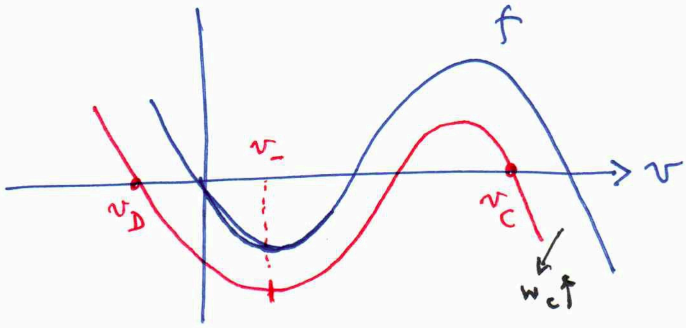
& this is certainly true for $v > v_-$ as shown

i.e for $v_c - V > v_-$

i.e $V < v_c - v_- = V_-$ say

note $\frac{\partial v_c}{\partial w_c} < 0$

so $\frac{\partial V_-}{\partial w_c} < 0$

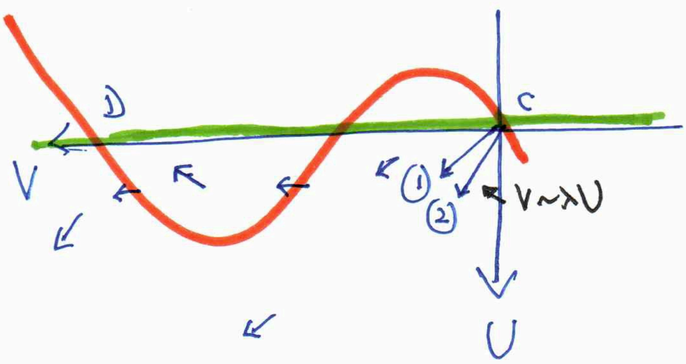


we have $\quad \frac{du}{dv} = c - \frac{[f(v) - w_c]}{u}$

i.e with $v = v_c - V$, $g = f - w_c$

$$-\frac{du}{dV} = c - \frac{g(V, w_c)}{u}$$

Let's write $U = -u$



$$\frac{dU}{dV} = c + \frac{g(V, w_c)}{U}$$

Near $U = V = 0$ $\quad g \simeq g' V = -f'(v_c) V \qquad \left(\text{have } \frac{df}{dv} = f'\right)$

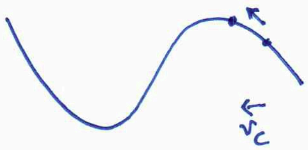
so $U \simeq \lambda V$ where $\quad \lambda = c - \frac{f'(v_c)}{\lambda}$

Thus $\lambda^2 - c\lambda + f'(v_c) = 0$

$$\lambda = \tfrac{1}{2}\left[c + \left(c^2 + 4|f'(v_c)|\right)^{1/2}\right] \qquad (f'(v_c) < 0)$$

want +ve root

So separatrix in $U, V > 0$ i.e $\lambda > 0$



as $w_c \uparrow$ $v_c \downarrow$ and $|f'(v_c)| \downarrow$

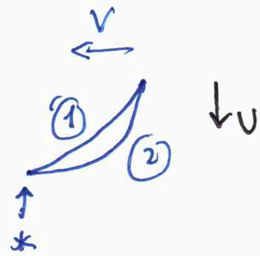
so $\frac{\partial \lambda}{\partial w_c} < 0$

Comparison argument.

Select two values of λ, λ_1 & λ_2 with $\lambda_1 < \lambda_2$ (see fig on p. 3)

i.e corresponding w_c values $w_1 > w_2$

For small V, $U_1 < U_2$.



This will remain true for $V > 0$ unless $\exists\, V^*$ s.t.

at $V = V^*$, $U_1 = U_2 = U^*$ and $U_2' \le U_1'$

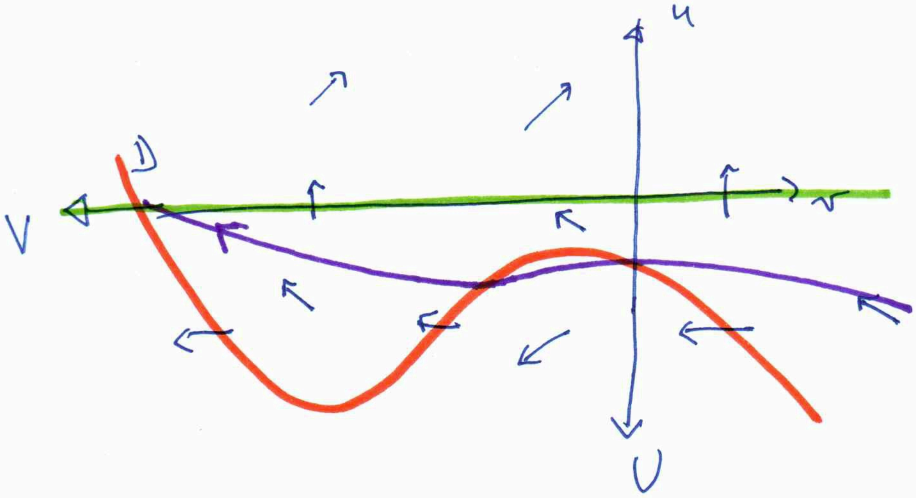
i.e $c + \frac{g(V^*, w_2)}{U^*} \le \frac{c + g(V^*, w_1)}{U^*}$

i.e $g(V^*, w_2) \le g(V^*, w_1)$ ie $\frac{g(V^*, w_1) - g(V^*, w_2)}{w_1 - w_2} \ge 0$

i.e $V < V_-(w_c)$

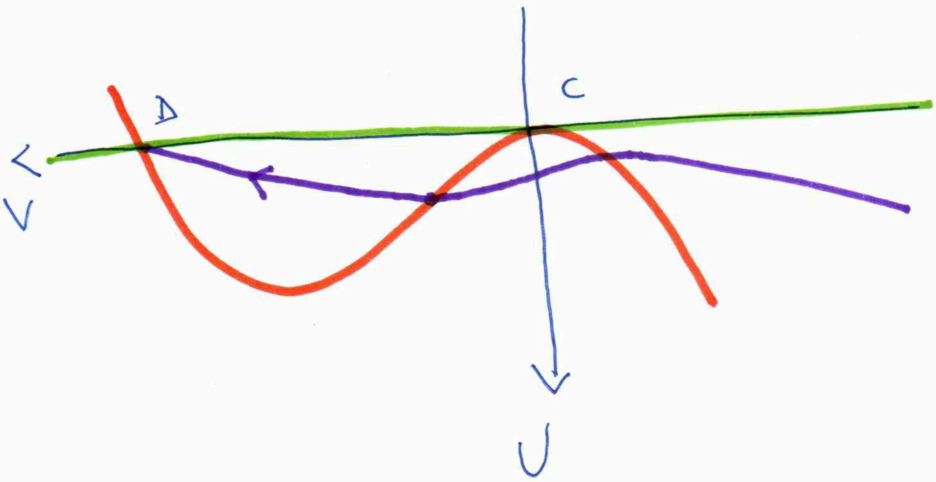
which requires $\frac{\partial g}{\partial w_c} \ge 0$. X. [for $\Rightarrow \frac{\partial U}{\partial w_c} < 0$ (at least for $V < v_-$) i.e $\frac{\partial u}{\partial w_c} > 0$

Next : if $w_c > w_M$ then phase plane is

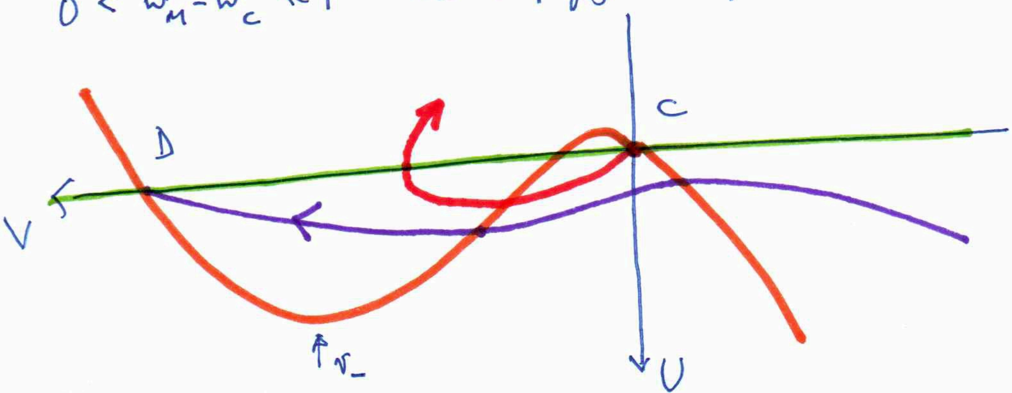


The separatrix to D may or may not cross the V nullcline as shown but clearly if $w_c > w_M$ it must remain in $u < 0$.

This is also true for $w_c = w_M$



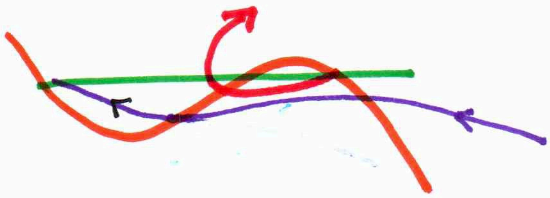
and since trajectories vary ~~smoothly~~ continuously with w_c we must have for $0 < w_M - w_c \ll 1$ $\Rightarrow$ implying the separatrix from C hits $u = 0$ before D ( )



The idea of the comparison argument is this & it works if

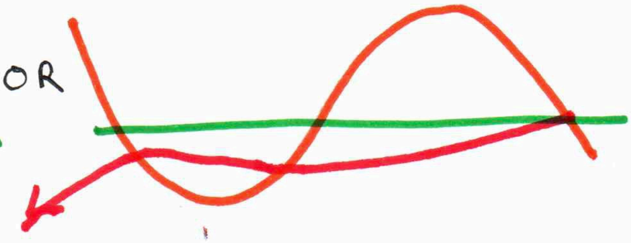
$\frac{\partial g}{\partial w_c} < 0$ everywhere

For w_c near w_M we have



For w_c near 0 we have

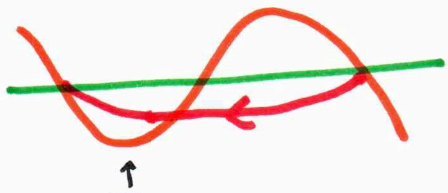
OR



Either way, continuous dependence of solutions of $\frac{du}{dv} = c - \frac{[f(u) - w_c]}{u}$

with $u \sim \lambda(v - v_c)$ near $v = v_c$ shows that there is at least one

value of w_c for which connection to $v = v_D$ occurs:



The uniqueness result follows from the comparison argument if $\frac{\partial g}{\partial w_c} < 0$

everywhere

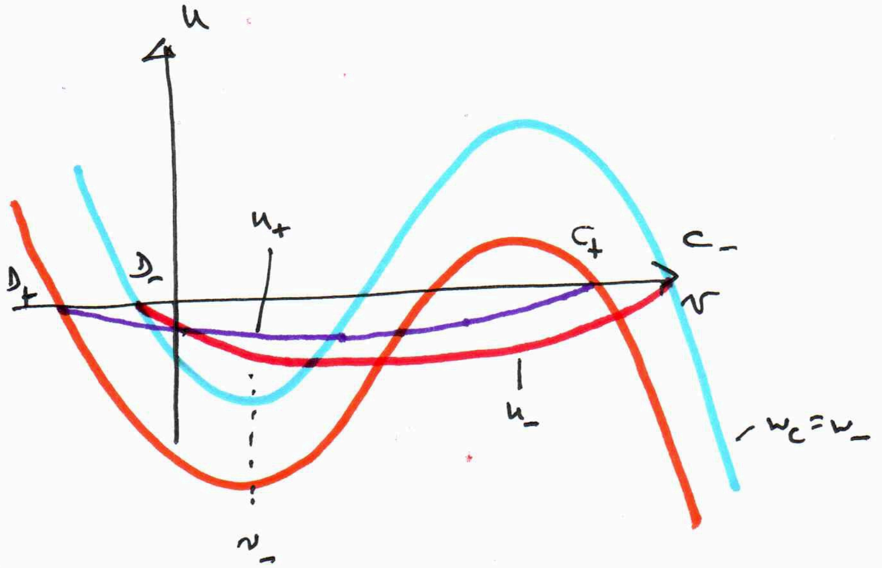
However, this is only true $\left(\frac{\partial g}{\partial w_c} < 0\right)$ if $V < V_-$ — so this doesn't work.

But...

We return to u & v and consider two values of w_c, $w_- < w_+$:

and assume that a heteroclinic connection (fixed pt to fixed pt)

occurs for both : the connecting trajectories are as shown $[u_\pm(v)]$



It is clear that u_+ & u_- must intersect, and where they do

we must have $\frac{du_+}{dv} \geq \frac{du_-}{dv}$ where $u_+ = u_-$ at v^* say.

Thus $$\frac{du_+}{dv} = c - \frac{[f(v) - w_+]}{u} \geq \frac{du_-}{dv} = c - \frac{[f(v) - w_-]}{u}$$

i.e. $\frac{w_+}{u} \geq \frac{w_-}{u} \Rightarrow w_+ \leq w_-$ (as $u < 0$) $\times$.

So in fact the solution is unique

$\square$